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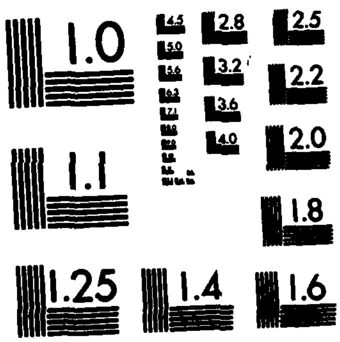

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An indirect numerical technique has been proposed to generate a rapid and accurate solution to a class of problems with linear state equations in which singular arcs occur. Then, by linearization this technique is extended to solve a class of problems with non-linear state equations. By modifying this technique a second approach has been obtained that can solve a broader class of singular problems with linear state equations.

The effect of the deviation of one player from the saddle point strategy on the performance index and the opponent's strategy has been studied for a two person zero-sum differential game with perfect information. An inverse system technique is used to determine the opponent's strategy by periodically measuring the state or the output of the system. Then, the proposed technique for singular problems is applied periodically to generate an approximate closed loop solution (which takes into consideration the deviation of the opponent from the saddle point trajectory) to achieve better performance than simply following the optimal open loop solution. A numerical example is presented to illustrate the efficiency of the proposed algorithm, and, comparisons have been made between the results of the open loop and closed loop solutions.

SINGULAR DIFFERENTIAL GAME NUMERICAL TECHNIQUE AND CLOSED LOOP GUIDANCE AND CONTROL STRAGETIES

By

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CHAPTER 1

INTRODUCTION. REVIEW AND CLASSIFICATION OF GAMES

SCOPE OF THE DISSERTATION

In the literature of zero-sum differential game theory a great deal of attention has been directed towards the pursuit-evasion games. This is due to applicability of this theory to military oriented problems or physical problems in areas such as engineering and economics. In many problems in these areas signularity is a source of difficulty for obtaining an exact solution. Singular arcs quite often occur in problems when one or more components of the controls appear linearly in the Hamiltonian function. Some effort is required to study differential games with singular arcs and to find an efficient technique to generate an accurate solution. Analytical solutions, in general, cannot be found except for linear quadratic and a few simple non-linear differential game problems. Therefore, this research is focused on the computational aspects of the singular problems. Our effort is to find an efficient technique with accuracy and rapid convergence which can be used for on-line purposes.

A common approach to the zero-sum differential game is to find a joint optimal solution for which each player assumes the other player is rational and uses his optimal strategy. Although this assumption basically is correct there are many reasons why a player may not be able to follow the exact saddle point solution. For example, bias error in the controller, a lack of accuracy in the computational technique or suboptimal playing of players will cause some deviation from the real optimal trajectories.

One way that a player can take partial advantage of the deviation of the opponent from the optimal open loop strategy is to measure the state of the game periodically and at each time of measurement apply the saddle point strategy assuming the other player plays optimally. It would be more desirable to find out the oppenent's non-optimal strategy (if it is possible) and consider this strategy as an additional known input to the system and solve a one-sided optimization problem. By this method one can gain more than he could by using his own optimal open loop policy.

In differential games the strategy of the opponent can be determined periodically through an inverse system if such a system exists and there is perfect information of the system states. By periodically solving an optimal control problem, using the proposed numerical technique an approximate closed-loop strategy dependent solution can be obtained for singular problems.

In order to put the present work in perspective we briefly review the highlights of the game theory in this chapter.

1.1 Review and Classification of Games

In the game theory there are different classification schemes. One of the proposed shcemes is as such:

Static Games, which are not associated with time, and

Dynamic Games, which have time evolution. All kinds of game problems are based on the concept of optimization theory.

Both static and dynamic optimization problems consist of three descriptive elements which were defined by Ho[11] in the framework of Generalized Control Theory. These are payoff functions or performance index, controller, and avaialable data to the controller. Once these elements are specified, further classifications can be made.

Static games can be continuous or discrete. In a continuous static game, the payoff function is expressed as a continuous algebraic relationship and there are infinite numbers of strategies for each player. The discrete static game is sometimes called a matrix or bimatrix game. In these types of games there are finite numbers of strategies for each player. If the sum of the payoffs to each of the players is zero, the game is a zero sum game, otherwise it is a non-zero-sum game. If the sum of the payoffs is constant the game is called a constant sum game.

The other major class of games is dynamic or differential games in which the state of the game evolves as a continuous function of time, and the strategies in these games are also continuous functions of time. In this type of game the system is generally expressed by a vector differential equation of the form:

$$\dot{x}(t) = f(x(t),u(t),v(t),t)$$

with a given initial state $x(t_0) = x_0$ and a scalar payoff function

$$J = h(x(t_f),t_f) + \int_{t_0}^{t_f} L(x(t),u(t),v(t),t)dt$$

when the state of the game evolves in discrete times and strategies are also implemented in discrete time, then, the game is called a discrete differential game. In this type of game the dynamic system is generally expressed by a vector difference equation of the form:

$$x_{K+1} = f(x_K, u_K, v_K, K)$$

with a given initial state $x(0) = x_0$ and a scalar payoff function

$$J = h(x_N, N) + \sum_{K=0}^{N-1} L(x_K, u_K, v_K, K)$$

Games can be categorized according to the information data to the controllers. When each player has perfect information about the state of the game, the system structure and the other player's payoff function, then, the game is called a perfect information game. If some of the states are not measurable in the game and/or parameters of the game are unknown (at least to one player) an imperfect information game results. The game is called stochastic game if at least one player has uncertain or random knowledge of some states of system's parameters.

Games can be classified according to the goal of the game.

Game of Kind or Qualitative Game. In this game the payoff function of the game is usually expressed in a discrete way as a win or loss. For example, the caputre occurs or does not occur in a pursuit-evasion game.

Game of Degree or Quantitative Game. In this type of game the payoff is usually a continuous functional. For example, in a pursuit-evasion game, one player tries to minimize and the other tries to maximize the separation distance at some final time. It is assumed that a capture does not occur during the game.

Strategies which are used in dynamic games can be classified as:

1. Open Loop Strategies, in which controls are functions of time and initial conditions, e.g., $u = u(t, x_0, t_0)$. The disadvantage of this kind of strategy in dynamic game problems is the inability to take advantage of non-optimally playing of the opponents. This strategy can be identified for the entire time of the game before the game starts.

2. Closed Loop Strategies, in which controls are functions of time and/or current state, e.g., $u = u(x,t)$. This kind of strategy can take partial advantage of non-optimally playing of the opponent, because it will be assumed that the opponent will play optimally from the time of measurement on.

In a differential game problem if a player chooses his strategy based on information of his opponent non-optimally playing strategy, he may be able to achieve a better performance than the closed loop solution. The strategy which is chosen in this manner is called closed-loop strategy dependent.

Pure and Mixed Strategies. If at each instant of time during the evolution of the game the values of the strategies are known the game is called pure strategy game. But if the strategies at each point of time are randomized and their characteristics are specified by probability density functions, the game is called mexed strategy game. Pure strategy games can be considered as a special case of mixed strategy games in which the probability of some particular controls are equal to one. Although in a static or a discrete and multistage game the concept of mixed strategy can be easily visualized, in continuous dynamic games the visualization is very difficult. Thus, it has not found much application in the analysis of differential games. In dynamic games pure or mixed strategies can be open loop or closed loop.

1.2 Types of Solutions

As was mentioned earlier in this chapter, games may be classified as zero-sum or non-zero sum games.

In zero sum games one player tries to minimize and the other player tries to maximize payoff functions. In such games a saddle point solution is sought such that each player optimizes his objective assuming the other player does the same. The resulting value of the payoff function is called the value of the game. The mathematical definition of the saddle point for both static and dynamic games is:

$$J(u^*,v) \leq J(u^*,v^*) \leq J(u,v^*)$$

where in a dynamic game u and v are functions of time and/or current states.

In non-zero-sum games there are different types of solutions, i.e., Nash equilibrium solutions, non-inferior solution (or parato optimal sets) and minmax solutions.

<u>Nash Solutions</u>. If u_i is the strategy for i^{th} player and $J_i(u_1,\ldots,u_N)$ is his cost functional, then a Nash solution u_i^* is defined by the following relationship:

$$J_i(u_1^*,\ldots,u_i^*,\ldots,u_N^*) \leq J_i(u_1^*,\ldots,u_i,\ldots,u_N^*)$$

where

$$u_i \neq u_i^* \qquad i = 1,\ldots,N$$

which implies that no player can improve his payoff by unilaterally deviating from his Nash solution, provided all other players use their Nash strategies. This solution has the characteristic of being protected against cheating; however, each player should play rationally, i.e., no player can attempt to increase another player's cost without regard to his own loss.

Note: When the game is zero sum, i.e., $J_1 = -J_2$ the Nash solution is the same as the saddle point solution.

<u>Noninferior Solutions</u>. It may be possible to achieve simultaneously a superior payoff than the Nash solution which was a non-cooperative solution, if certain cooperation among the players is made in a prescribed manner.

This kind of solution is used in classical game theory and modern welfare economics. In this set of solutions no player can achieve a better pay-off unless it is at the expense of the other player.

Min-Max Solution. This kind of solution represents a "security level" for each player. The min-max solution u_i^* for the i^{th} player is the strategy which satisfies

$$J_i(u_i^*) = \min_{u_i} \max_{u_J} J_i(u_1,\ldots,u_N) \qquad \forall\, J \neq i$$

This relationship implies that the other players try to do the worst damage to the i^{th} player and the i^{th} player tries to gain the most.

1.3 Review of Methods of Solutions for Differential Games

The study of differential games was initiated by Isaacs in 1954. His approach was formal and closely resembled the dynamic programming approach to optimization problems. In 1964 Berkowitz and Fleming applied a calculus of variation technique to a simple differential game. Later on Berkowitz treated a wider class of differential games by the same technique. More recently functional analysis has been applied to differential game problems as a rigorous approach, and certain highly mathematicsl problems without direct physical interpretation have been solved by this approach (Freedman[2]).

Geometric approach is another rigorous and interesting technique to differential games which provides some insight into the problems and has been used by some authors (Balaquier, Gerald, Lietman[58]).

In 1969 Bryson and Ho[4] treated a class of zero-sum-linear quadratic differential games and by the application of the set of necessary conditions for the saddle point. They obtained a set of closed loop solutions. Then by forming some auxiliary problems they verified the existence of the saddle point. In 1970, McFarland used the same class of problems and, without the assumption of a saddle point used another approach to show the existence of the saddle point. Although the obtained solutions for this class of problems are analytical, however, the solution to the same class of problems with control and/or state constraints generally cannot be analytical(17).

In order to solve problems not having analytical solutions, some numerical techniques have been proposed to generate optimal open loop solutions. In these techniques it has been assumed that the saddle point solution exists and singularity does not occur. Several techniques which are used in optimal control problems have been applied to differential game problems, including neighboring optimal techniques (which are closely associated with successive sweep method), quasilinearization, and differential dynamic programming.

There are a few closed loop techniques that have been proposed for generating a near optimal solution. Anderson(40,41) has worked on an updating technique for generating a near optimal closed loop solution to a zero sum perfect information differential game by periodically updating the solution to the two point boundary value problem obtained by the application of the necessary condition for the saddle point solution. Jachimowitz(47) has proposed an adaptive technique, based on estimation theory, to determine the non-optimal strategy of the opponent through the state measurement. Then by converting the game to a one-sided optimal control problem he generated a near optimal closed loop solution for non-singular differential games. Behn and Ho[56] and Gonzalez[48] have used inverse systems to determine the opponent strategy for the solution of non-singular stochastic and deterministic zero sum differential games.

The review of the literature in this area shows that the computational aspect of singular game problems requires special attention and it is the subject of this dissertation.

1.4 Objective and Scope of the Dissertation

It is the main purpose of this dissertation to present a technique that generates an accurate and rapid optimal open loop solution to a class of singular differential game problems.

In the next chapter some definitions and theorems which are common between sigmular optimal control theory and singular differential game theory have been introduced. These results are very useful in providing insight to the form of the singular solutions of the problems. In this chapter we define a new class of two-person zero-sum differential games (generalized pursuit-evasion game) with pure strategy and perfect information, which have linear state equations. In this problem the pursuer's and evader's controls are respectively bounded and unbounded and the performance index is quadratic in terms of the state and the evader's control. The final time is fixed and the game is considered as a game of degree.

Some conditions for the strict convexity and strict concavity of the performance index with respect to the pursuer's and the evader's controls are derived to guarantee the existence of a unique saddle point solution in this class of problems.

In Chapter 3 an indirect numerical technique with two approaches are offered. The first approach can only solve problems with singular arcs. In this technique the sequence of controls in the entire time interval of the game are estimated. The solution to the set of two point boundary value problems (obtained from the necessary conditions for optimality) is generated by an iterative procedure using Newton's method. This approach iterates only on switching times between control arcs and has at least quadratic convergence. This technique is extended to a class of non-linear differential games by linearization.

The second approach which is somehow similar to the first approach can also solve linear bang band and highly dimensional problems. The solution to the set of TPBVP is obtained by iterating on the initial costates and switching times between control arcs.

Numerical results of a physical example are reported in this chapter.

Chapter 4 discusses closed loop on line solutions for this class of differential games. The concept of the inverse system is introduced. The existence of the inverse system is discussed and an algorithm is used which incorporates the necessary and sufficient condition for its existence. The proposed algorithm, together with the inverse system is applied to generate an approximate closed-loop strategy dependent solution. Computational comparisons have been made between the saddle point solutions and the cases that one player deviates from the saddle point strategy, and the other plays open loop and closed loop strategies.

Chapter 5 summarizes all the results obtained in this report. Advantages and disadvantages of numerical techniques are discussed. Some areas of work for future research are also recommended.

CHAPTER 2

SINGULAR OPTIMAL CONTROL AND DIFFERENTIAL GAME PROBLEMS

Singular arcs may arise in many optimal control problems. In the past two decades singularity has received considerable theoretical attention. This problem was defined by Rozonoer (1959) and has been studied by Johnson and Gibson (1963), Robbins (1966), Goh (1966), McDonell and Powers (1970). The authors have developed some necessary conditions and also McDonell and Powers have obtained sufficient conditions for optimal control assuming there exists a totally singular extremal. Singular solutions in optimal control have been thought by many people to be of only academic interest. However, singular arcs quite often appear in engineering, economics and chemical problems. Siebenthal and Aris (1964) have shown that optimal singular arcs occur in chemical reactor startup problems. Optimal trajectories of mass-varying vehicles which are subjected to aerodynamic forces include singular arcs. The sounding rocket problem[4], Saturn Guidance singular flat earth[22] and resource allocation problems are other examples of singularities. In the following section we will formulate an optimal control problem and discuss the possibility of occurrence of singular arcs and later on we will extend this subject to differential game problems.

2.1 Problem Formulation

The fundamental problem of optimal control theory can be formulated in equivalent forms of Bolza, Mayer and Lagrange. The Bolza problem is the following:

Find the control function $u(\cdot)$ which minimizes (or maximizes) the performance functional

$$J = h(x(t_f),t_f) + \int_{t_0}^{t_f} L(x,u,t)dt \tag{2.1-1}$$

subject to:

$$\dot{x}(t) = f(x(t),u(t),t) \tag{2.1-2}$$

$$x(t_0) = x_0 \text{ given} \tag{2.1-3}$$

$u(\cdot)$ belongs to the set U and t is a member of $[t_0,t_f]$,

x is an n dimensional state vector. u is an m dimensional control vector, h and L are scalar functions and are assumed smooth. The control set is defined by

$$U \triangleq \{u(\cdot):u_i(\cdot) \text{ is piecewise continuous in time, } |u_i(\cdot)| < \infty,$$
$$t_0 \leq t \leq t_f, \quad i = 1,2,\ldots,m\} \tag{2.1-4}$$

The initial state and initial time are specified in the final state and final time may be specified or unspecified.

As a prerequisite to solving this problem by the maximum principle we define the Hamiltonian as

$$H(x,u,\lambda,t) = L(x,u,t) + \lambda^T f(x,u,t) \tag{2.1-5}$$

Necessary conditions for u^* to be an optimal control are

$$\dot{x}^*(t) = \frac{\partial H}{\partial \lambda}(x^*(t),u^*(t),t) \tag{2.1-6}$$

$$\dot{\lambda}^*(t) = \frac{\partial H}{\partial x}(x^*(t),u^*(t),\lambda^*(t),t) \tag{2.1-7}$$

$$H(x^*(t),u^*(t),\lambda^*(t),t) \le H(x^*(t),u(t),\lambda^*(t),t) \tag{2.1-8}$$

for all admissible $u(t)$ and for all $t \epsilon [t_0,t_f]$ and boundary conditions

$$\left[\frac{\partial h}{\partial x}(x^*(t_f),t_f) - \lambda^*(t_f)\right]^T \delta x_f + \Big[H(x^*(t_f),u^*(t_f),\lambda^*(t_f),t_f) + \frac{\partial h}{\partial t}(x(t_f),t_f)\Big]\delta t_f = 0 \tag{2.1-9}$$

where λ is an n dimensional Lagrangian multiplier vector. The optimal control u should satisfy (2.1-8) and usually extremal u is obtained as

$$u^* = \arg\min_u H(x,u,\lambda,t) \tag{2.1-10}$$

and in the case that there are bounds on the control such that

$$|u_i(t)| \le K_i(t) \qquad i = 1,2,\ldots,m \tag{2.1-11}$$

and the Hamiltonian is linear in terms of u_i assuming the components of the control are independent, the extremal controls can be expressed as

$$u_i(t) = K_i(t)\operatorname{sgn} Hu_i \quad \text{if} \quad Hu_i(t) \neq 0 \quad \text{for} \quad t \in [t_1,t_2] \qquad (2.1\text{-}12)$$

where $[t_1,t_2]$ belongs to time interval $[t_0,t_f]$

$$\operatorname{sgn} Hu_i = \begin{cases} +1 & \text{if } Hu_i < 0 \\ -1 & \text{if } Hu_i > 0 \end{cases} \qquad (2.1\text{-}13)$$

Note that there may exist conditions such that $Hu_i = 0$ for some non-zero time interval. Then we will have problems of singularity. With respect to this more general problem some basic definitions and theorems of singular optimal control problems are introduced.

Definition 2.1. If one or more components of the control function u appear linearly in the Hamiltonian, and there exists a non-zero time interval $[t_1,t_2]$ in $[t_0,t_f]$ such that the coefficient of at least one of these components are zero on $[t_1,t_2]$. Then the control is said to be singular. In this interval maximum principle (2.1-8) provides no information about the control u^* and its relationship with state and costate x^* and λ^*.

Definition 2.2. Let u_i be the ith element of the optimal singular control vector u on the interval $[t_1,t_2]$ belonging to $[t_0,t_f]$, which appears linearly in the Hamiltonian. Let $2q$ be the lowest order of the time derivative of Hu_i in which u_i appears explicitly with a coefficient which is not identically zero on the subinterval of $[t_1,t_2]$. Then q is called the order of the singular subarc.

Definition 2.3. Assuming all the components $u_1, u_2, \ldots, u_m$ of the control vector u are singular simultaneously, then u is called a totally singular control function when

$$\frac{\partial H}{\partial u}(x, \lambda, t) = 0 \quad \text{for} \quad t \in [t_0, t_f] \tag{2.1-14}$$

Definition 2.4. If (2.1-14) holds for arcs in K subintervals of length T_i, $i = 1, 2, \ldots, K$ such that

$$\sum_{i=1}^{K} T_i < t_f - t_0 \tag{2.1-15}$$

then the problem is called partially singular.

So, for the existence of the singular arc it is necessary that the Hamiltonian be a linear function of at least one component of a control vector. The analysis of such problems is complicated by the fact that the solution in general consists of some combination of singular and non-singular subarcs. The number and sequence of these subarcs are not known a priori, and it is almost impossible to establish the existence of singular arcs without actual numerical solutions.

The following theorem which is a necessary condition for the optimality of singular subarcs is due to Robbins(31) and junction theorems which are followed are given by McDonell and Powers(21).

Theorem 2.1 (Generalized Legendre-Clebsch Condition). On an optimal singular subarc of order q, it is necessary that

$$(-1)^q \frac{\partial}{\partial u}\left[\frac{d^{2q}}{dt^{2q}} \frac{\partial H}{\partial u}\right] \geq 0 \tag{2.1-16}$$

and if only inequality holds it is called strengthened GLC.

The essence of the proof is given in Appendix A and the complete proof is found in Reference (31).

Since d^{2q} / dt^{2q} Hu is the lowest time derivative of Hu in which control u appears explicitly in the general form, we can have

$$\frac{d^{2q}}{dt^{2q}} Hu(x,\lambda,t) = A(x(t),\lambda(t),t) + B(x(t),\lambda(t),t)u_s \tag{2.1-17}$$

A and B as a function of time are defined as

$$\alpha(t) \equiv A(x(t),\lambda(t),t) \tag{2.1-18}$$

$$\beta(t) \equiv B(x(t),\lambda(t),t) \tag{2.1-19}$$

The above notations are used in the proof of the theorems in this chapter.

2.2 <u>The Junctions Theorems</u>. Although the analysis of totally singular control problems are rather well developed, in partially singular control problems the analysis of junction points are not yet fully understood. Since a useful sufficient condition for such problems is not available, one has to study the necessary conditions which are valid in the neighborhood of a junction between singular and non-singular subarcs. It is expected that such conditions can be used to eliminate candidate extremals or predict beforehand the way in which singular and non-singular subarcs must be joined or whether the optimal control is continuous or discontinuous at a junction point.

Assuming the optimal control is well-behaved in a neighborhood of a junction, then the following theorem holds.

Theorem 2.2. Let t_s be a point at which singular and non-singular arcs of an optimal control u are joined, and let q be the order of singular subarcs. Suppose the strengthened GLC condition is satisfied, and assume that the control is piecewise analytic in a neighborhood of t_s. Let $u^{(r)}$ $(r \geq 0)$ be the lowest order time derivative of u which is discontinuous at t_s. Then $q + r$ is an odd integer (proof is given in Appendix A).

Two corollaries follow from Theorem 2.2.

Corollary 1. In q even problems, assuming u is piecewise analytic, and the strengthened GLC condition is satisfied, then the optimal control is continuous at each junction.

Corollary 2. In q odd problems, assuming u is piecewise analytic, and the strengthened GLC condition is satisfied, then the optimal control either has a jump discontinuity at each junction or else the singular control joins the boundary smoothly, i.e. with a continuous first derivative.

For Theorem 2.2 strengthened GLC conditions should be satisfied at the junction point t_s. There is also a possibility that the GLC condition be satisfied with equality at this point. If q is the order of the singular arc $\frac{\partial}{\partial u}H_u^{(2q)} = \beta(t)$ cannot be identically zero on the singular subarc. Therefore, in view of analyticity assumptions, a derivative of some order of $\beta(t)$ must be non-zero at time t_s even if $\beta(t_s) = 0$. This leads to the generalization of the Theorem 2.2 which is stated as a separate theorem to emphasize the important results of Theorem 2.2.

<u>Theorem 2.3</u>. Let t_s be a point at which singular and non-singular subarcs of an optimal control u are joined, and let q be the order of singular arc. Assume that the control is piecewise analytic in a neighborhood of t_s, and let $\beta^{(m)}$ $(m \geq 0)$ be the lowest order derivative of the GLC expression $\frac{\partial}{\partial u} Hu^{(2q)} \equiv \beta$ which is non-zero at t_s, then

1. if $m \leq r$, $q + r + m$ is an odd integer

2. if $m > r$, $-\text{sgn}[\beta^{(m)}(t_s^+)\beta^m(t_s^-)] = (-1)^{q+r+m}$

The proof of this theorem is similar to that for Theorem 2.2.

The conclusion of this theorem is that if $m > r$, $\beta^{(m)}$ may not be continuous at junction t_s and if $m \leq r$, $\beta^{(m)}$ is continuous at junction point t_s.

Theorems 2.2 and 2.3 require the assumption of piecewise analyticity of the control in a neighborhood of the junction.

This hypothesis is usually satisfied on the singular subarc, but not always on the nonsingular subarc. Thus, we are led to consider properties which do not require the assumption of analyticity as stated in the following theorem.

Theorem 2.4. Let u be an optimal control which contains both non-singular subarcs and piecewise continuous q^{th} order singular subarcs.

1. If $Hu^{(2q)} \neq 0$ on the non-singular side of a junction, then the control is discontinuous.
2. If $A \equiv 0$, $B \neq 0$ and $K \neq 0$ at a junction then, the control is discontinuous.
3. If u is piecewise continuous on the non-singular subarc d^{2q} / dt^{2q} Hu = 0 on the non-singular side of a junction, and $B \neq 0$ at the junction then the control is continuous.

Proof. For Case 1) knowing that d^{2q} / dt^{2q} Hu $\equiv 0$ on the singular subarc and d^{2q} / dt^{2q} Hu $\neq 0$ on the side of nonsingular subarc we have

$$\alpha(t_s) \pm \beta(t_s)K(t_s) \neq 0 = \alpha(t_s) + \beta(t_s)u_s(t_s)$$

From this relationship we obtain $|u_s(t_s)| \neq K(t_s)$. Therefore u is discontinuous.

For Case 2) $\alpha(t_s) = 0$ and $\beta(t_s) \neq 0$ imply $u_s(t_s) = 0$, and since $K(t_s) \neq 0$, the control is discontinuous.

For Case 3) since d^{2q} / dt^{2q} Hu = 0 for both singular and non-singular subarcs, we will have

$$\alpha(t_s) + \beta(t_s)u_n(t_s) = 0 = \alpha(t_s) + \beta(t_s)u_s(t_s)$$

and since $\beta(t_s) \neq 0$, then, $u_n(t_s) = u_s(t_s)$. Therefore the control is continuous at the junction.

In order to see how the singular subarc may occur in a problem we consider a simple scalar example.

2.3 Example of a Singular Control Problem . Find control u to minimize

$$J = \frac{1}{2}\int_0^{t_f} x^2(t)dt \tag{2.3-1}$$

subject to

$$\dot{x}(t) = u(t) \qquad x(t_0) = x_0 \tag{2.3-2}$$

given

$$|u| \leq 1 \tag{2.3-3}$$

where t_f is fixed final time.

The Hamiltonian is defined

$$H = \frac{1}{2}x^2 + \lambda u \tag{2.3-4}$$

and the set of necessary and boundary conditions are

$$\dot{x}^* = u^* \tag{2.3-5}$$

$$\dot{\lambda}^* = -x^* \tag{2.3-6}$$

$$\lambda^*(t_f) = 0 \qquad x(0) = x_0 \tag{2.3-7}$$

The optimal control u^* is obtained as the following

$$u = \begin{cases} +1 & \text{if } \lambda(t) < 0 \\ -1 & \text{if } \lambda(t) > 0 \\ \text{undetermined} & \text{if } \lambda(t) = 0 \end{cases}$$

If $\lambda(t) \equiv 0$ we will have

$$\dot{\lambda}(t) = 0 \Rightarrow x(t) = 0 \Rightarrow \dot{x}(t) = 0 \Rightarrow u_s(t) = 0 \qquad (2.3\text{-}9)$$

Also the strengthened GLC condition

$$(-1) \frac{\partial}{\partial u} \ddot{H}_u = 1 > 0 \qquad (2.3\text{-}10)$$

for singular arcs is satsified.

Now by changing the values of initial condition, final time and final state, we will consider several different cases.

Case 1. Let $x(0) = 1$, $t_f = 1$ and $x(1)$ be free, then the solution will be

$$\left.\begin{aligned} u^*(t) &= -1 \\ x^*(t) &= -t + 1 \\ \lambda^*(t) &= -\frac{1}{2}t^2 + t + \frac{1}{2} \end{aligned}\right\} \text{for } t \in [0,1] \qquad (2.3\text{-}11)$$

The optimal control and trajectories are shown in Figures (2.1-a) and (2.1-b).

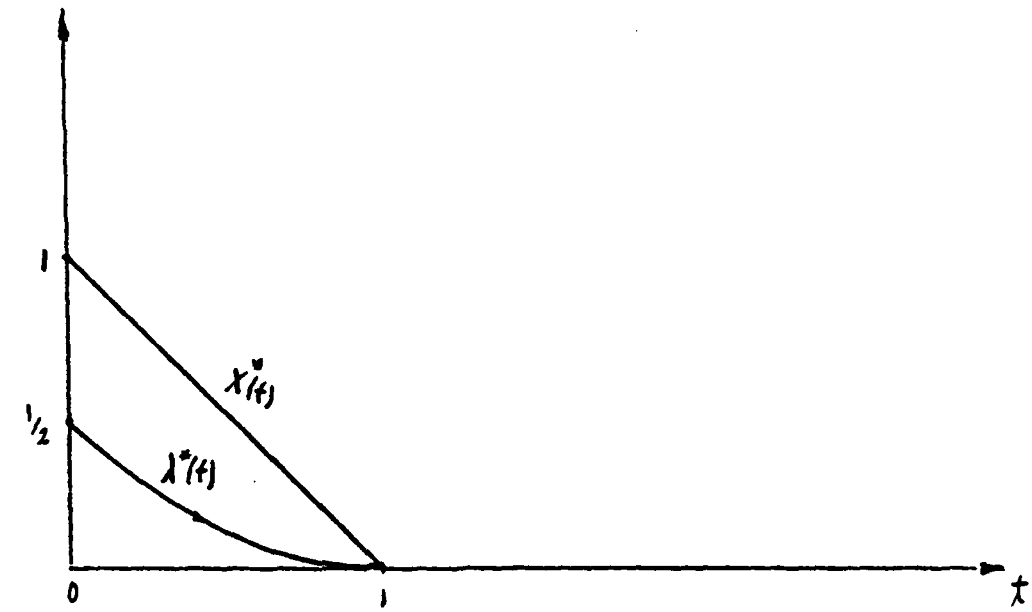

Figure 2.1-a. State and costate trajectories in a nonsingular solution.

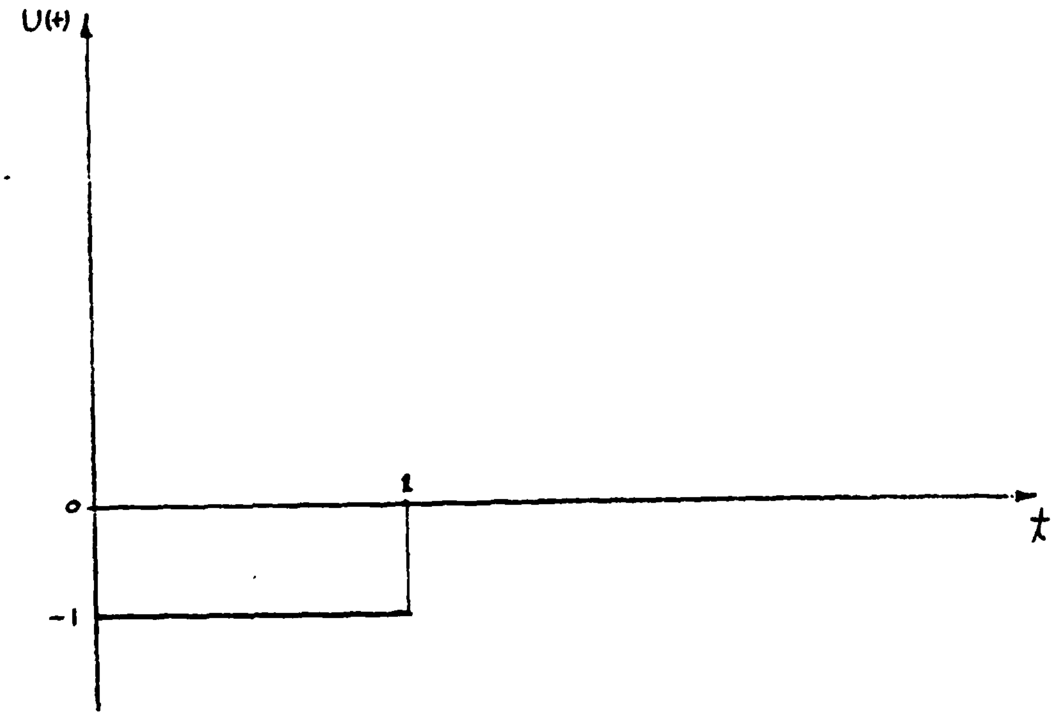

Figure 2.1-b. Nonsingular optimal control.

We see that with this initial state and final time singularity does not occur in any interval of the problem.

Case 2. Let $x(0) = 1$, $t_f = 2$ and $x(2)$ be free, then the solution is obtained as

$$\left.\begin{aligned} u^*(t) &= -1 \\ x^*(t) &= -t + 1 \\ \lambda^*(t) &= \frac{1}{2}t^2 - t + \frac{1}{2} \end{aligned}\right\} \text{for } t \in [0,1] \tag{2.3-12}$$

and

$$u^*(t) = 0 \quad \text{for} \quad 1 < t \leq 2 \tag{2.3-13}$$

$$\left.\begin{aligned} x^*(t) &= 0 \\ \lambda^*(t) &= 0 \end{aligned}\right\} \quad \text{for } t \in [1,2] \tag{2.3-14}$$

The optimal control and trajectories are shown in Figure 2.2-a and 2.2-b.

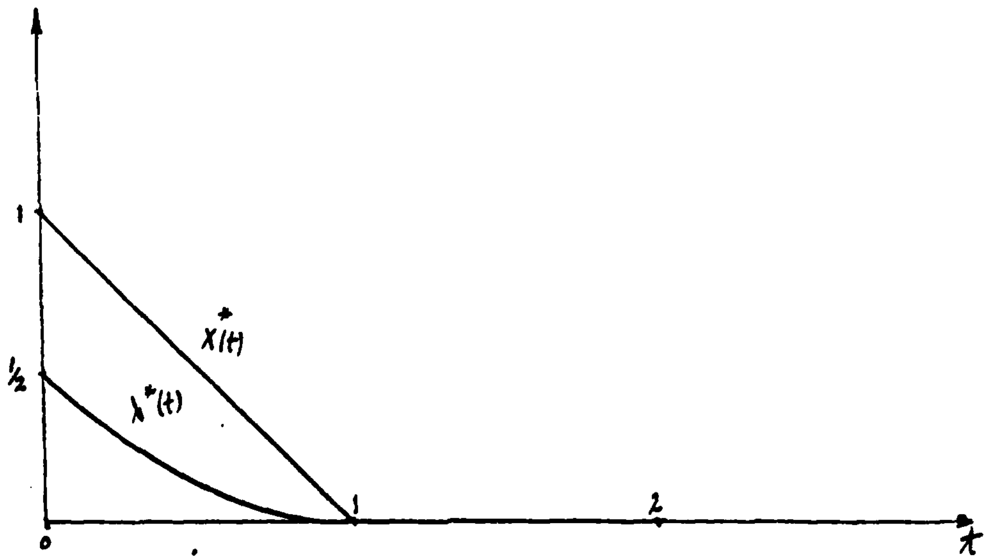

Figure 2.2-a. State and costate trajectories in nonsingular and singular intervals.

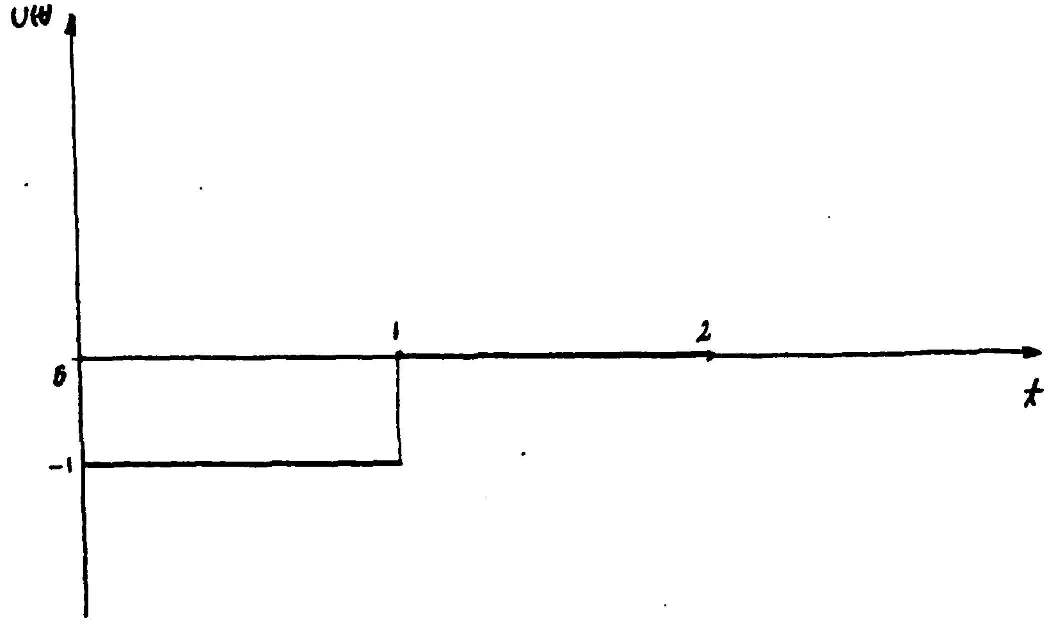

Figure 2.2-b. Nonsingular and singular controls.

It is noticed that when the final time in Case 1 is increased from 1 to 2, and initial state is the same and final state remains free, singular arc occurs in the interval $t \in [1,2]$.

Case 3. Let $x(0) = 1$, $t_f = 2$ and $x(2) = 1$. The solution will be

$$\left.\begin{aligned} u^*(t) &= -1 \\ x^*(t) &= -t + 1 \\ \lambda^*(t) &= \frac{1}{2}t^2 + t - \frac{1}{2} \end{aligned}\right\} \quad t \in [0,1] \tag{2.3-15}$$

$$u^*(t) = +1 \qquad 1 < t \leq 2 \tag{2.3-16}$$

$$\left.\begin{aligned} x^*(t) &= t - 1 \\ \lambda^*(t) &= -\frac{1}{2}t^2 + t - \frac{1}{2} \end{aligned}\right\} \quad t \in [1,2] \tag{2.3-17}$$

The optimal control and trajectories are shown in Figures 2.3-a and 2.3-b.

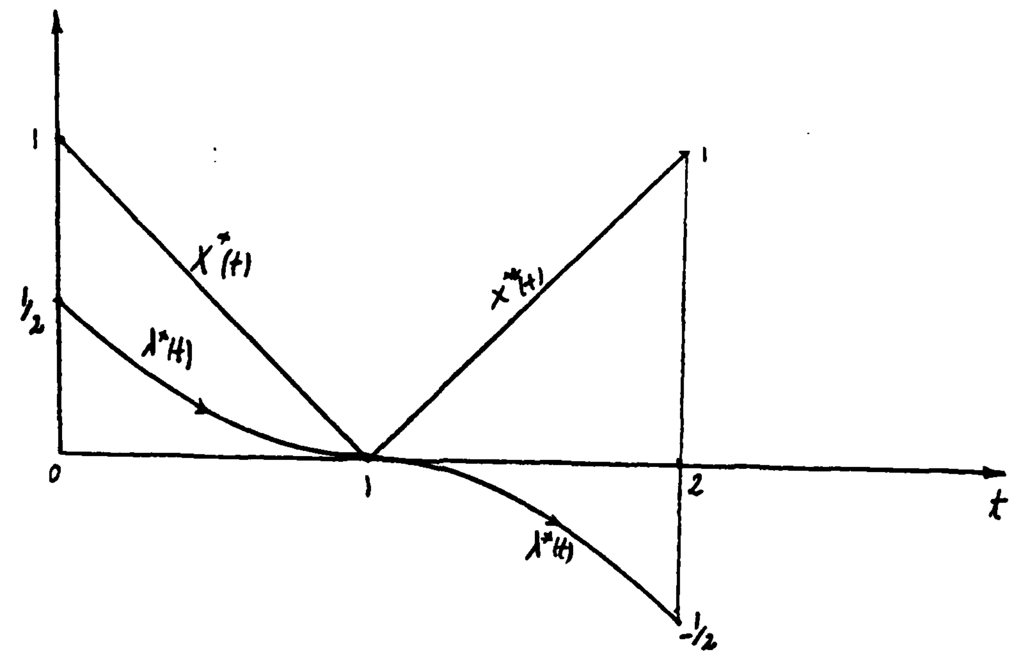

Figure 2.3-a. State and costate trajectories (nonsingular solution)

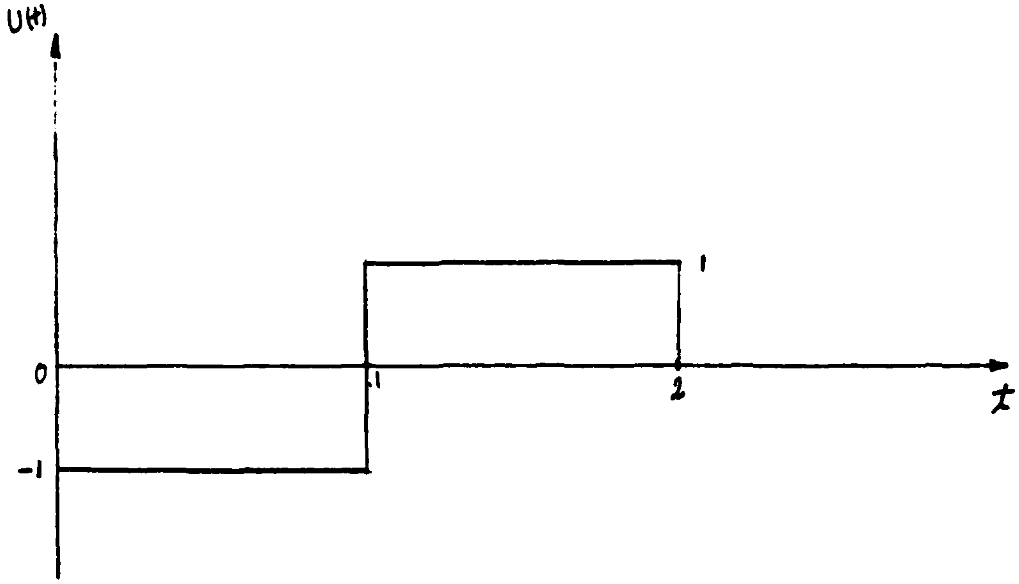

Figure 2.3-b. Bang-Bang Control.

In this case we have fixed the final state. This causes the solution of Case 2 to change from a singular solution to a bang-bang solution.

<u>Case 4</u>. Now we modify Case 3 by increasing the final time from $t_f = 2$ to $t_f = 3$.

The solution is obtained as

$$\left.\begin{aligned} u^*(t) &= -1 \\ x^*(t) &= -t + 1 \\ \lambda^*(t) &= \frac{1}{2}t^2 - t + \frac{1}{2} \end{aligned}\right\} \quad t \in [0,1] \qquad (2.3\text{-}18)$$

$$u^*(t) = 0 \qquad 0 < t < 2 \qquad (2.3\text{-}19)$$

$$\left.\begin{aligned} x^*(t) &= 0 \\ \lambda^*(t) &= 0 \end{aligned}\right\} \quad t \in [1,2] \qquad (2.3\text{-}20)$$

$$\left.\begin{aligned} u^*(t) &= 1 \\ x^*(t) &= t - 2 \\ \lambda^*(t) &= -\frac{1}{2}t^2 + 2t - 2 \end{aligned}\right\} \quad t \in [2,3] \qquad (2.3\text{-}21)$$

The optimal control and trajectories are shown in Figures 2.4-a and 2.4-b.

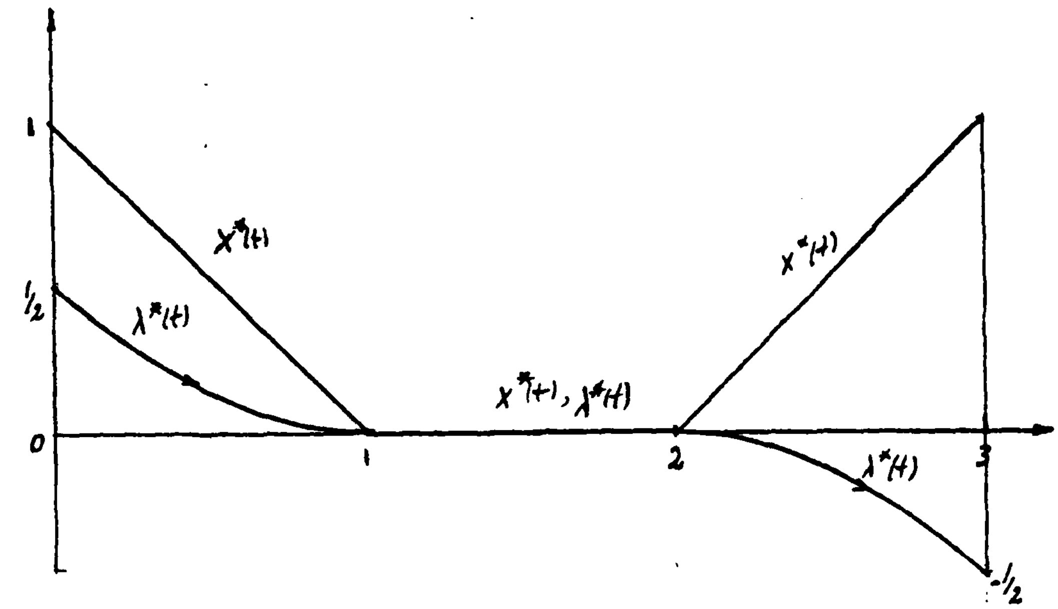

Figure 2.4-a. State and costate trajectories with singular interval.

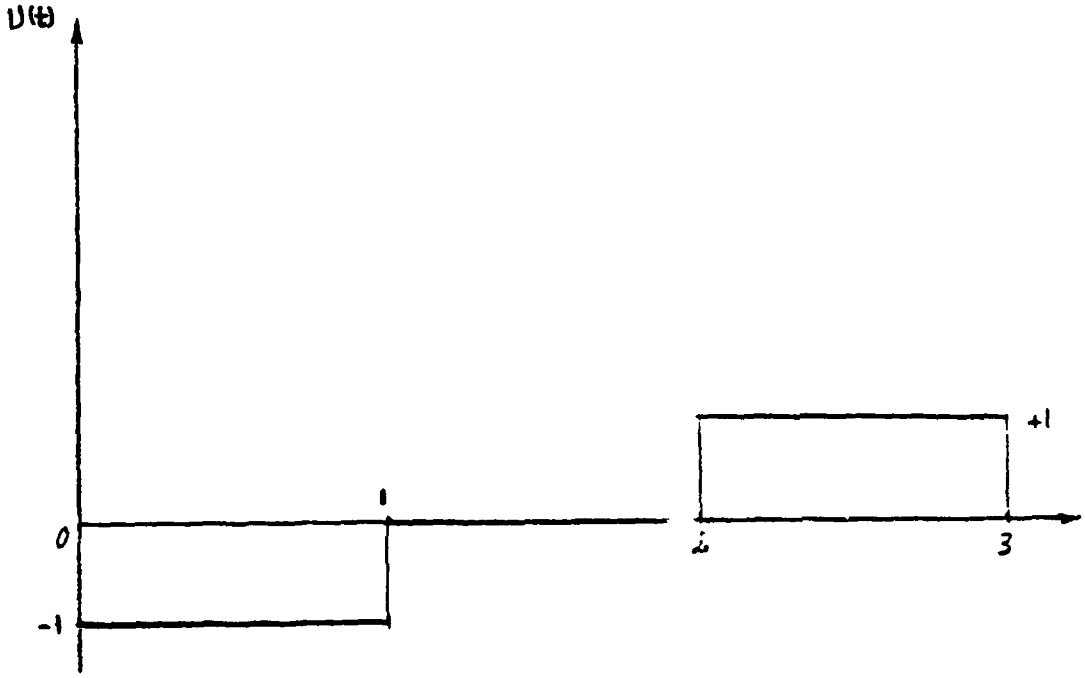

Figure 2.4-b. Bang-singular-bang control.

It is noticed that in Case 3, by changing the final time, the solution of the problem changes from bang-bang to a solution with singular arc. Indeed there are many simple changes which can be made to the effect that singularity will occur or be removed. For example, in Case 2, if the initial condition is increased from $x(0) = 1$ to $x(0) = 2$ with the same time duration, singular control will be removed from the solution and the control solution will stay on the lower bound until the final time is reached.

Singular arcs have also appeared in differential game problems. The Homicidal Chauffeur game[1] is a pursuit-evasion game with the possibility of singular arc. Problems of Thrust-limited rockets subjected to aerodynamic forces are another example of pursuit-evasion games with the possibility of singular or intermediate thrust arcs for either or both pursuer and evader. In the next section we formulate a differential game problem and consider cases with singular arcs.

2.4 Singular Differential Game Problem

A Bolza type differential game problem can be formulated as the following

$$J(u,v) = h(x(t_f),t_f) + \int_{t_0}^{t_f} L(x,u,v,t)dt \tag{2.4-1}$$

subject to

$$\dot{x}(t) = f(x,u,v,t) \tag{2.4-2}$$

$$x(t_0) = x_0 \text{ given.} \tag{2.4-3}$$

There are two players, (P) is trying to minimize J by control u and (E) is trying to maximize J by control v. It is assumed that both players have perfect information about the system, and also each player has partial control over the game.

The problem is to determine u^* and v^* such that

$$J(u^*,v) \leq J(u^*,v^*) \leq J(u,v^*) \tag{2.4-4}$$

holds. The solutions specified by u^* and v^* are optimal and termed the saddle point solution.

$u(\cdot)$ and $v(\cdot)$ respectively belong to the sets U and V. The final time is assumed to be fixed. h and L are scalar functions and assumed to be smooth. The control sets are defined by

$$U \triangleq \{u(\cdot):u_i \text{ is piecewise continuous in time } |u_i(\cdot)| < \infty$$

$$t_0 \leq t \leq t_f \quad i = 1,2,\ldots,m\} \tag{2.4-5}$$

$$V \triangleq \{v(\cdot):v_j(\cdot) \text{ is piecewise continuous in time } |v_j(\cdot)| < \infty$$

$$t_0 \leq t \leq t_f \quad j = 1,2,\ldots,r\} \tag{2.4-6}$$

the initial state and initial time are fixed and final state is free.

In order to find the saddle point, the Hamiltonian is defined as

$$H(x,u;v,t) = L(x,u,v,t) + \lambda^T f(x,u,v,t) \tag{2.4-8}$$

where $\lambda \in R^n$ is Lagrangian multiplier.

Assuming there exists a saddle point solution, the set of necessary conditions for the saddle point solutions are

$$\dot{x}^*(t) = f(x^*,u^*,v^*,t) \qquad x(t_0) = x_0 \tag{2.4-9}$$

$$\dot{\lambda}^*(t) = -\frac{\partial H}{\partial x} = -H_x \tag{2.4-10}$$

$$H(x^*,\lambda^*,u^*,v,t) \le H(x^*,\lambda^*,u^*,v^*,t) \le H(x^*,\lambda^*,u,v^*,t) \tag{2.4-11}$$

$$\lambda(t_f) = \frac{\partial h(x(t_f),t_f)}{\partial x} \tag{2.4-12}$$

The saddle point solution should satisfy (2.4-11) and usually is obtained as

$$u^* = \arg\min_u H(x^*,u,v^*,\lambda^*,t) \tag{2.4-13}$$

$$v^* = \arg\max_v H(x^*,u^*,v,\lambda^*,t) \tag{2.4-14}$$

In the case that Hamiltonian is linear in terms of components of v and u and there are bounds on controls, i.e.

$$|u_i| \le K_{1i} \qquad i = 1,2,\ldots,m \tag{2.4-15}$$

$$|v_j| \le K_{2j} \qquad j = 1,2,\ldots,r \tag{2.4-16}$$

optimal controls can be expressed as:

$$\begin{cases} u_i = -K_{1i}\,\mathrm{sgn}\, Hu_i(x^*,\lambda^*,u,v^*,t) & \text{if } Hu_i \neq 0 \\ & i = 1,2,\ldots,m \\ v_j = K_{2j}\,\mathrm{sgn}\, Hv_j(x^*,\lambda^*,u^*,v,t) & \text{if } Hv_j \neq 0 \\ & j = 1,2,\ldots,r \end{cases} \quad (2.4\text{-}17)$$

assuming all components of u and v are independent of each other.

Note that similar to the optimal control problems there may exist conditions such that Hu and Hv or both be equal to zero for some non-zero time interval in the game. Then we will face the problem of singularity.

Definition 2.5. If one or more components of the control functions u or v or both appear linearly in the Hamiltonian defined in (2.4-8) and there exists a non-zero subinterval of time $[t_1,t_2]$ between t_0 and t_f such that the coefficient of at least one of the control components is zero on this subinterval. The control is said to be singular and this subinterval is said to be a singular interval. The maximum principle (2.4-11) provides no information about these controls and their relationship with x^* and λ^* in this interval.

All the definitions in Section 2.1 about the singularity in optimal control problems will hold for singular two-sided problems and Anderson[38] has derived necessary conditions for optimality of singular arcs in differential games which are exactly the same as necessary conditions for optimality of singular control problems. Thus, the analog of the GLC condition in the two-sided problems are

$$(-1)^q \frac{\partial}{\partial u}\left[\frac{d^{2q}}{dt^{2q}} H_u\right] \geq 0 \qquad (2.4\text{-}18)$$

$$(-1)^q \frac{\partial}{\partial u}\left[\frac{d^{2q}}{dt^{2q}} H_v\right] \leq 0 \qquad (2.4\text{-}19)$$

and the analysis of junction points carries over as well.

2.5 Derivation of Singular Control. For simplicity of calcuation a restricted class of nonlinear singular differential games is considered.

Find the saddle point solution u^* and v^* to the payoff

$$J = h(x(t_f)) \qquad (2.5\text{-}1)$$

subject to:

$$\dot{x}(t) = f_1(x)u + f_2(x,v) \qquad (2.5\text{-}2)$$

$$x(t_0) = x_0 \text{ given} \qquad (2.5\text{-}3)$$

$$|u| \leq K \qquad (2.5\text{-}4)$$

where $x \in R^n$, $v \in R^r$, u is scalar, f_1 and f_2 are $n \times 1$ vector value functions at least n times differentiable with respect to x, h is a smooth scalar function, and t_f is fixed. Assiming the saddle point solution exists, the set of necessary conditions are obtained as the following.

Define the Hamiotonian:

$$H(x,\lambda,u,v) = \lambda^T f_1(x)u + \lambda^T f_2(x,v) \qquad (2.5\text{-}5)$$

since v is not required to be bounded. We form

$$\frac{\partial H}{\partial v} = 0 \qquad (2.5\text{-}6)$$

and solve for the control v^*. (Assuming v can be expressed explicitly in terms of x and λ)

$$v^* = v(x,\lambda) \qquad (2.5\text{-}7)$$

substituting (2.5-7) into the state and costate equations we will have

$$x(t) = f_1(x)u + f_2(x,v,(x,\lambda)) \qquad (2.5\text{-}8)$$

$$\dot{\lambda}(t) = -f_{1x}^T u - f_{2x}^T \lambda \qquad (2.5\text{-}9)$$

where $f_{1x} = \frac{\partial f_1(x)}{\partial x}$ and $f_{2x} = \frac{\partial f_2(x,v)}{\partial x}$.

For a nonsingular interval

$$u = -K \operatorname{sgn} Hu \qquad Hu \neq 0 \qquad (2.5\text{-}10)$$

and for a singular interval

$$\frac{\partial H}{\partial u} = 0 \qquad (2.5\text{-}11)$$

To find the singular control in this interval we consider (2.5-11) and its respective time derivatives

$$\frac{\partial H}{\partial u}(x,\lambda) = 0 \Rightarrow \lambda^T f_1(x) = 0 \qquad (2.5\text{-}12)$$

as it is noticed this relationship does not yield any information about singular control u_s. So we take the time derivative of (2.5-12) substituting from (2.5-8) and (2.5-9) for x and λ and we have

$$\frac{d}{dt}\frac{\partial H}{\partial u}(x,\lambda) = 0 \Rightarrow \lambda^T(f_{1x}f_2 - f_{2x}f_1) = 0 \qquad (2.5\text{-}13)$$

(2.5-13) still does not yield any information about singular control u_s.

Let

$$\begin{cases} g(x) = f_{1x}f_2 - f_{2x}f_1 \\ \dfrac{\partial g(x)}{\partial x} = g_x \end{cases} \qquad (2.5\text{-}14)$$

$$\frac{d^2}{dt^2}\frac{\partial H}{\partial u}(x,\lambda) = \dot{\lambda}^T g_x + \lambda^T \dot{g} = \lambda^T(g_x f_1 - f_{1x} g)u$$

$$+ \lambda^T(g_x f_2 - f_{2x} g) = 0 \qquad (2.5\text{-}15)$$

If $\lambda^T(g_x f_1 - f_{1x} g) \neq 0$ the singular control

$$u_s(x,\lambda) = \frac{\lambda^T(g_x f_2 - f_{2x} g)}{\lambda^T(g_x f_1 - f_{1x} g)} \qquad (2.5\text{-}16)$$

If

$$\lambda^T(g_x f_1 - f_{1x} g) = 0 \qquad (2.5\text{-}17)$$

we continue taking time derivatives of (2.5-17) until u appears explicitly with a nonzero coefficient. In most porblems of interest u appears in the n^{th} order time derivative of (2.5-12). If this control is optimal then it satisfies the strengthened GLC condition.

2.6 Linear Quadratic Singular Differential Game with Bound on Control.

A class of generalized linear pursuit-evasion differential game is described by the following state equations and performance index

$$J = \frac{1}{2} x(t_f)Sx(t_f) + \frac{1}{2}\int_{t_0}^{t_f} (x^TQx - v^TRv)dt \qquad (2.6\text{-}1)$$

subject to:

$$\dot{x}(t) = Ax + Bu + Cv \qquad (2.6\text{-}2)$$

$$x(t_0) = x_0 \quad \text{given} \qquad (2.6\text{-}3)$$

$$|u| \le K \qquad (2.6\text{-}4)$$

where

x $n \times 1$ state vector

u scalar

v $r \times 1$ vector

A, B, C are continuous time varying matrices with compatible sizes.

S $n \times n$ continuous time varying positive semidefinite matrix.

Q $n \times n$ continuous time varying positive definite matrix.

R $r \times r$ continuous time varying positive definite bounded symmetric matrix.

t_f fixed final time.

In this class of problems there are two players. Pursuer, (P) seeks control u to minimize J, and evader (E) tries to find control over his own state.

In the next section we will show that this problem under some conditions possesses a unique saddle point solution.

2.6.1. Saddle Point Conditions. We now impose some conditions on the matrices A, B, C, Q and R to assure the existence of the saddle point.

In this class of linear quadratic differential game for the existence of a unique saddle point, the performance index should be strictly convex with respect to u for any fixed v and strictly concave with respect to v for any fixed m.

Strict convexity of J with respect to u is easily verified by the given assumption of positive definiteness of Q and controllability of the system (AB) and since u belongs to a set of convex, closed and bounded there will be a unique minimum u^* for any given v.

To establish concavity of J with respect to v some conditions are required so that concavity of the second term dominates convexity of the first term of the integrand.

First assuming $S \equiv 0$, the solution to (2.6-2) is given by

$$x(t) = \varphi(t,t_0)x(t_0) + \int_{t_0}^{t} \varphi(t,\tau)\beta(\tau)u(\tau)d\tau + \int_{t_0}^{t} \varphi(t,\tau)C(\tau)v(\tau)d\tau \tag{2.6-5}$$

Assuming u is fixed and substituting from (2.6-5) into (2.6-1) and considering just those terms which are nonlinear in v we will get

$$J'(v) = \int_{t_0}^{t_f} \left(\int_0^t \varphi(t,\tau)C(\tau)v(\tau)d\tau\right)^T Q\left(\int_0^t \varphi(t,\tau)C(\tau)v(\tau)d\tau\right)dt - \int_{t_0}^{t_f} v(t)Rv(t)dt \tag{2.6-6}$$

For strict concavity of J with respect to v assuming (AC) is controllable it will be sufficient to show

$$J'(v) < 0 \tag{2.6-7}$$

$v(t) \in V$ such that $\int_{t_0}^{t_f} |v(t)|^2 dt < \infty$.

Define

$$w(t) = R^{-1/2}\, v(t) \qquad \forall\, t \in [t_0,t_f] \tag{2.6-8}$$

$$\|w(t)\|_1 \triangleq \left(\sum_{i=1}^{r} |w_i(t)|^2\right)^{1/2} \tag{2.6-9}$$

$$\|w\|_2 \triangleq \left(\int_{t}^{t_f} \|w(t)\|_1^2 \, dt \right)^{1/2} \tag{2.6-10}$$

Satisfaction of (2.6-7) is equivalent to showing that

$$\int_{t_0}^{t_f} \left(\int_0^t \varphi(t,\tau)c(\tau)R^{-1/2} w(\tau)d\tau\right)^T Q\left(\int_{t_0}^{t} \varphi(t,\tau)c(\tau)R^{-1/2} w(\tau)d\tau\right)dt - \int_{t_0}^{t_f} w^T(\tau)w(\tau)d\tau < 0 \tag{2.6-11}$$

We normalize $w(\tau)$ by defining

$$\xi(\tau) = \frac{w(\tau)}{\|w\|} \tag{2.6-12}$$

so

$$\|\xi\| = 1 \tag{2.6-13}$$

Assuming $\|w\|_2 \neq 0$ and dividing (2.6-11) by $\|w\|_2^2$ and using the relationship (2.6-12) it is enough to show

$$\int_{t_0}^{t_f} \left(\int_{t_0}^{t} \varphi(t,\tau)c(\tau)R^{-1/2} \xi(\tau)d\tau\right) Q\left(\int_{t_0}^{t} \varphi(t,\tau)c(\tau)R^{-1/2} \xi(\tau)d\tau\right)dt < 1 \tag{2.6-14}$$

Using norm inequalities

$$\left| \int_{t_0}^{t_f} \left(\int_{t_0}^{t} \varphi(t,\tau)c(\tau)R^{-1/2}\, \xi(\tau)d\tau\right)^T Q\left(\int_{t_0}^{t} \varphi(t,\tau)c(\tau)R^{-1/2}\, \xi(\tau)d\tau\right)dt \right|$$

$$\leq \int_{t_0}^{t_f} \left(\int_{t_0}^{t} \|\varphi(t,\tau)c(\tau)R^{-1/2}\|_1^2\ \|\xi(\tau)\|_1\ \|Q\|_1\ t\ d\tau\right)dt$$

$$\leq \int_{t_0}^{t_f} \left(\int_{t_0}^{t} \|\varphi(t,\tau)c(\tau)R^{-1/2}\|_1^2 d\tau \int_{t_0}^{t} \|\xi(\tau)\|_1^2 d\tau \|Q\|_1\right)t\ dt$$

$$= \int_{t_0}^{t_f} \left(\int_{t_0}^{t} \|\varphi(t,\tau)c(\tau)R^{-1/2}\|_1^2 d\tau \|Q\|_1\right)t\ dt \tag{2.6-15}$$

Therefore

$$\sup_{\|\xi\|=1} \left| \int_{t_0}^{t} \left(\int_{t_0}^{t} \varphi(t,\tau)c(\tau)R^{-1/2}\xi(\tau)d\tau\right)^T Q\left(\int_{t_0}^{t} \varphi(t,\tau)c(\tau)R^{-1/2}\xi(\tau)d\tau\right)dt \right|$$

$$\leq \int_{t_0}^{t_f} (t - t_0) \int_{t_0}^{t} \|\varphi(t,\tau)c(\tau)R^{-1/2}\|_1^2 d\tau \|Q\|_1 dt \tag{2.6-16}$$

where

$$\|Q\|_1 = \sup_{\|x\|_1=1} \|QX\|_1 \tag{2.6-17}$$

So if

$$\boxed{\int_{t_0}^{t_f} (t - t_0) \int_{t_0}^{t} \|\varphi(t,\tau)c(\tau)R^{-1/2}\|_1^2 d\tau \|Q\|_1 dt < 1} \tag{2.6-18}$$

Then

$$\int_{t_0}^{t_f} \left(\int_{t_0}^{t} \varphi(t,\tau)c(\tau)v(\tau)d\tau\right)^T Q\left(\int_{t_0}^{t} \varphi(t,\tau)c(\tau)v(\tau)d\tau\right)dt$$

$$- \int_{t_0}^{t_f} v^T(t)Rv(t)dt < 0 \qquad (2.6\text{-}19)$$

Note condition (2.6-18) implies the negative definiteness of J with respect to v which easily results in the concavity of J with respect to v.

Besides (2.6-18) we must show that J is radially unbounded in v in order to establish the existence of saddle points.

It can be verified that $J(u,v) \to \infty$ as $\|v\| \to \infty$ for any fixed u by the following relationships

$$\int_{t_0}^{t_f} \left(\int_{t_0}^{t} \varphi(t,\tau)c(\tau)v(\tau)d\tau\right)^T Q\left(\int_{t_0}^{t} \varphi(t,\tau)c(\tau)v(\tau)d\tau\right)dt$$

$$\leq \int_{t_0}^{t_f} (t - t_0) \int_{t_0}^{t} \|\varphi(t,\tau)c(\tau)R^{-1/2}\|_1^2 d\tau \|Q\|_1 dt \|w\|_2^2$$

$$(2.6\text{-}20)$$

and

$$\int_{t_0}^{t_f} v(t)Rv(t)dt = \|w\|_2^2 \qquad (2.6\text{-}21)$$

Subtracting (2.6-20) from (2.6-19) we will obtain

$$J'(v) \leq \left(1 - \int_{t_0}^{t_f} \int_{t_0}^{t} \|\varphi(t,\tau)c(\tau)R^{-1/2}\|_1^2 d\tau \|Q\|_1 dt\right) \|w\|^2 \qquad (2.6\text{-}22)$$

From (2.6-18), the above parenthesis is a positive constant number for any fixed t_f. Since R is bounded $\|v\| \to \infty$ implies that $\|w\|_2 \to \infty$ and from $J'(v) \le -K\|w\|_2^2$, $K > 0$ $J'(v) \to \infty$. So it is concluded that $J(u,v)$ is radially unbounded for any fixed u.

Since for this class of problems all the required conditions for the existence of a saddle point hold, so (2.6-18) is a sufficient condition for the existence of a saddle point.

In a more general case when $S \neq 0$ the term outside the integral of the performance index will yield a similar term

$$(t_f - t_0) \int_{t_0}^{t_f} \|\varphi(t_f,\tau)c(\tau)R^{-1/2}\|_1^2 \, \|S\|_1 d\tau \qquad (2.6\text{-}23)$$

which is added to the left side of (2.6-18).

2.6.2. Necessary Conditions. Assuming there exist a saddle point solution, define Hamiltonian as

$$H = \frac{1}{2} x^T Qx - \frac{1}{2} v^T Rv + \lambda^T Ax + \lambda^T Bu + \lambda^T cv \qquad (2.6\text{-}24)$$

where λ is an $n \times 1$ Lagrangian multiplier vector.

The set of necessary conditions are

$$\dot{x}^* = Ax^* + Bu^* + Cv^* \qquad x(t_0) = x_0 \qquad (2.6\text{-}25)$$

$$\dot{\lambda}^* = -Qx^* - A^T\lambda^* \qquad (2.6\text{-}26)$$

$$\frac{\partial H}{\partial v} = Hv = Rv^* - c^T\lambda = 0 \qquad (2.6\text{-}27)$$

$$H(x^*,\lambda^*,u^*,v) \le H(x^*,\lambda^*,u^*,v^*) \le H(x^*,\lambda^*,u,v^*) \qquad (2.6\text{-}28)$$

$$\lambda^*(t_f) = Sx^*(t_f) \qquad (2.6\text{-}29)$$

The strengthened GLC condition for the optimality of the singular control of the first order

$$\frac{\partial}{\partial u}\frac{d^2}{dt^2} Hu = B^TQB > 0 \qquad (2.6\text{-}30)$$

is satisfied.

From (2.6-27) control v^* is obtained as

$$v^* = -R^{-1}C^T\lambda^* \qquad (2.6\text{-}31)$$

Substituting for v from (2.6-31) in the state equations, the set of two point boundary value problem will be

$$\begin{cases} \dot{x}(t) = Ax + Bu - CR^{-1}C^T\lambda & (2.6\text{-}32) \\ \dot{\lambda}(t) = -Qx - A^T\lambda & (2.6\text{-}33) \\ x(t_0) = x_0 & (2.6\text{-}34) \\ \lambda(t_f) = Sx(t_f) & (2.6\text{-}35) \end{cases}$$

The control u is obtained as

$$u = \begin{cases} +K & \text{if } B^T\lambda < 0 \quad \text{for } t \in [t_1, t_2] \\ -K & \text{if } B^T\lambda > 0 \quad \text{for } t_1 \in [t_1, t_2] \\ \text{undetermined} & \text{if } B^T\lambda = 0 \quad \text{for } t_1 < t \le t_2 \end{cases}$$

(2.6-36)

As has been mentioned in the interval of time that $\partial H(x,\lambda,u)/\partial u = B^T\lambda = 0$, the control is singular and obtained by taking successive time derivatives of $\partial H/\partial u$ until u_s explicitly appears. To calculate singular control $u_s(x,\lambda)$ we find

$$\frac{\partial H}{\partial u} = B^T\lambda = 0 \qquad t_1 < t \le t_2 \tag{2.6-37}$$

$$\frac{d}{dt}\frac{\partial H}{\partial u} = \dot{B}^T\lambda + B^T\dot{\lambda} = -B^TQx + (B^T - B^TA)\lambda = 0 \quad \text{for } t_1 < t \le t_2$$

(2.6-38)

Continue taking time derivatives of (2.6-38) since $B^TQB > 0$ then the singular control appears in the next relationship so that

$$u_s = Mx + N\lambda \qquad t_1 < t \le t_2 \tag{2.6-39}$$

where

$$M = (B^TQB)^{-1}(B^TAQ - \dot{B}^TQ + B^TQ - \dot{B}^Tq - B^TQA) \tag{2.6-41}$$

$$N = (B^TQB)^{-1}(B^T\dot{Q}cR^{-1}c^T + \ddot{B}^T\dot{B}^TA - B^T\dot{A}^T - \dot{B}^TA + B^TAA^T) \tag{2.6-42}$$

By substitution of non-singular and singular controls in the state equations we will obtain

$$\begin{cases} \dot{x}(t) = Ax - CR^{-1}C^T\lambda \pm BK & (2.6\text{-}43) \\ \dot{\lambda}(t) = -Qx - A^T\lambda & (2.6\text{-}44) \end{cases}$$

which hold on non-singular interval and

$$\begin{cases} \dot{x}(t) = (A + BM)x + (BN - CR^{-1}C^T)\lambda & (2.6\text{-}45) \\ \dot{\lambda}(t) = - Qx - A^T\lambda & (2.6\text{-}46) \end{cases}$$

which hold on singular intervals. In Chapter 3 we will show how to treat these sets of equations with given and obtained boundary conditions as a multipoint boundary value problem.

Remark. According to Reference (4) the existence of an optimal v can be verified through an auxiliary problem assuming $u^*(t)$ is an optimal open loop solution

$$\max_{v} \{\frac{1}{2} x^T(t_f)Sx(t_f) + \int_{t_0}^{t_f} (x^TQx - v^TRv)dt\} \tag{2.7-47}$$

subject to:

$$\dot{x}(t) = Ax + Bu + Cv \tag{2.6-48}$$

and

$$u = u^*(t) \quad \text{a time function} \tag{2.6-49}$$

The solution for v is

$$v(t) = R^{-1}C^T(Kx + s) \tag{2.6-50}$$

where

$$\dot{s}(t) + (KCR^{-1}C^T + A^T)s + KBu^* \qquad s(t_f) = 0 \tag{2.6-51}$$

$$\dot{K}(t) + A^TK + KA + KCR^{-1}C^TK + Q = 0 \quad (K(t_f) = S \tag{2.6-52}$$

Since $u^*(t)$ is bounded from the above we conclude that there will be an optimal v^* if there exists a finite solution to the Ricatti equation (2.6-52) for $0 < t < t_f$. In such a way the existence of a saddle point to the game (2.1-1) - (2.1-4) can be verified since there exists a unique u^* in this problem.

CHAPTER 3

THE COMPUTATION OF SINGULAR OPTIMAL CONTROL AND DIFFERENTIAL GAMES

Despite the amount of interest in singular control problems development of the computational aspect of this problem requires more attention. Due to the control bounds and control discontinuities some computational and analytical difficulties are encountered. To deal with these kinds of difficulties some special considerations should be taken in singular problems.

Pagurek and Woodside[28] (1968) have presented a direct method and applied a conjugate gradient method in function space for optimal control problems with bounds on controls and have solved a problem with singular arcs. In this technique the region of saturation should be guessed a priori and some procedure is devised to improve this guess at each iteration. Although the rate of convergence for some problems has been almost good, the solution obtained by this technique is not accurate. Also junction points between saturated and unsaturated controls are not obtained in their exact locations. Ko and Stevens[33] (1971) applied gradient methods to determine the optimal heat transfer coefficient distribution along a tubular reactor. This method handles both singular and bang bang arcs, but it obtains an approximate solution and in the presence of singular arcs has a very slow rate of convergence when it gets close to the optimum. Jacobson et al. have transformed the singular optimal control problem to a non-singular problem by adding a quadratic integral function of control to the performance index. This integral function is multiplied by a coefficient which tends to zero iteratively during the computation of non-singular control, so that, the solution converges to the original singular one. Jacobson, Gershwin and Lele[19] (1970) have applied differential dynamic programming (Jacobsen 1968[60]) to the resulting non-singular problem and have solved several examples. This technique has given some satisfactory results, but often computational difficulties arise when the coefficient of integral tends to zero. In the same reference a variant of the method called epsilon-aplha (ε-α) algorithm, has been proposed by the authors to overcome this difficulty. With this alternative method another parameter α is considered and the coefficient of the integral does not have to approach to zero. However, it still does not guarantee the convergence of the singular optimal control and accurate results cannot be obtained. In 1972 Edgar and Lapidus[35] used the method of Jacobson et al, together with differential dynamic programming and penalty function, and applied that to discretized versions of the problem. First they developed the algorithm for linear control problems, then they extended the technique to non-linear problems. This technique has the same computational difficulties as that of the method of Jacobson et al. The degree of accuracy is low, especially since the problem is descritized, the exact switching time cannot be obtained. Also this method is time consuming for non-linear and some linear problems. This technique has the advantage of capability of handling problems with high dimensions. Anderson[27] (1972) proposed an indirect method for computations of singular optimal control problems with first orders of singularity. This technique iterates on initial costates and a prescribed Terminal Error Function is minimized with respect to initial costates by a search technique. The sequence of non-singular and singular controls are estimated a priori and at each iteration switching conditions are checked and adjusted. The accuracy of this technique is good and the rate of convergence of the method depends on the rate of convergence of the search technique.

Yeo[37] (1975) applied quasilinearization together with $(\varepsilon-\alpha)$ algorithms of Jacobson to solve singular problems. This method can have a quick rate of convergence if the coefficient of the added quadratic integral is too small. But, as this coefficient gets smaller the likelihood of convergence gets smaller too. This method has a simple programming and faster computational speed than the same method using differential dynamic programming. In 1976 Edge and Powers[25] in a function space quasi-Newton algorithm applied Davidon and Broyden algorithms to optimal control problems with bounded and singular arcs. Examples indicated that this method is more accurate than gradient and conjugate gradients, but still the obtained solutions in an approximation of the real optimal solutions. In 1973 Aly[29] and Chan applied a modified quasilinearization technique to totally singular problems and in 1978 Aly[30] applied the same technique to partially singular problems with first order singularity. In this technique the sequence of controls are estimated a priori and at each iteration the initial costates are adjusted to satisfy the junction conditions. This adjustment may accelerate or may decelerate the rate of convergence of the modified quasilinearization used in this method.

In the following section we have presented an indirect method with two approaches. The first approach solves certain groups of problems which can provide n conditions at the junction of singular and non-singular controls. In these cases we solve a set of multipoint boundary value problems by iterating only on switching time. This approach has been proposed for linear problems, thus it is extended to nonlinear problems. The second approach can solve linear singular problems with singularities of order of one to n. In this approach an iterative procedure which iterates on switching time and initial costates is used. This method can directly solve singular differential game problems.

3.1 Numerical Techniques

In totally singular problems, since there is no discontinuity of the control and junction points, solutions to the set of TPBVP obtained from the necessary conditions for the saddle point may be obtained by some of the numerical techniques in the literature. In partially singular problems special considerations should be taken for the junction conditions and discontinuity of the control. So far, there has not been found a general sufficient condition for optimality of the partially singular control. If there exists a single solution to the problem satisfaction of the necessary conditions is enough for the optimality. But if there are a finite number of solutions, some comparison should be made in order to find the optimal one[34]. In the proposed technique we estimate the sequence of singular and non-singular controls in time intervals of the game. In the simplest case we consider only one junction point. For this case a finite number of possible sequences may occur. We consider two major different cases.

Case 1. Game starts with a non-singular arc and at some time t_s switches to a singular arc and terminates on the singular arc at time t_f.

Case 2. Game starts with a control on one bound and then at some time t_s the control switches to the other bound which is the bang bang case. For dealing with junction points, it is enough to consider these two cases and other cases are basically the same, but more tedious when the number of switchings are increased.

In Case 1 since control u is bounded, discontinuity of u will not cause discontinuity in the states and costates, however, it may create some corner points at the junction of singular and non-singular arcs. So, the following relationships hold at each junction point

$$x(t_s^-) = x(t_s^+) \tag{3.1-1}$$

$$\lambda(t_s^-) = \lambda(t_s^+) \tag{3.1-2}$$

where t_s^- and t_s^+ respectively represent time just before and time just after the switching time ts. Figures (3.1-a) - (3.1-d) are a possible scheme of controls, states and costates for a differential game.

The following relationship holds at any point along

$$\begin{aligned}
&\partial H/\partial u(x(t),\lambda(t),t) = 0 \\
&\frac{d}{dt}\frac{\partial H}{\partial u}(x(t),\lambda(t),t) = 0 \qquad t_s \le t \le t_f \qquad (3.1\text{-}3) \\
&\vdots \\
&\frac{d^{n-1}}{dt^{n-1}}\frac{\partial H}{\partial u}(x(t),\lambda(t),t) = 0
\end{aligned}$$

Schematic Figures

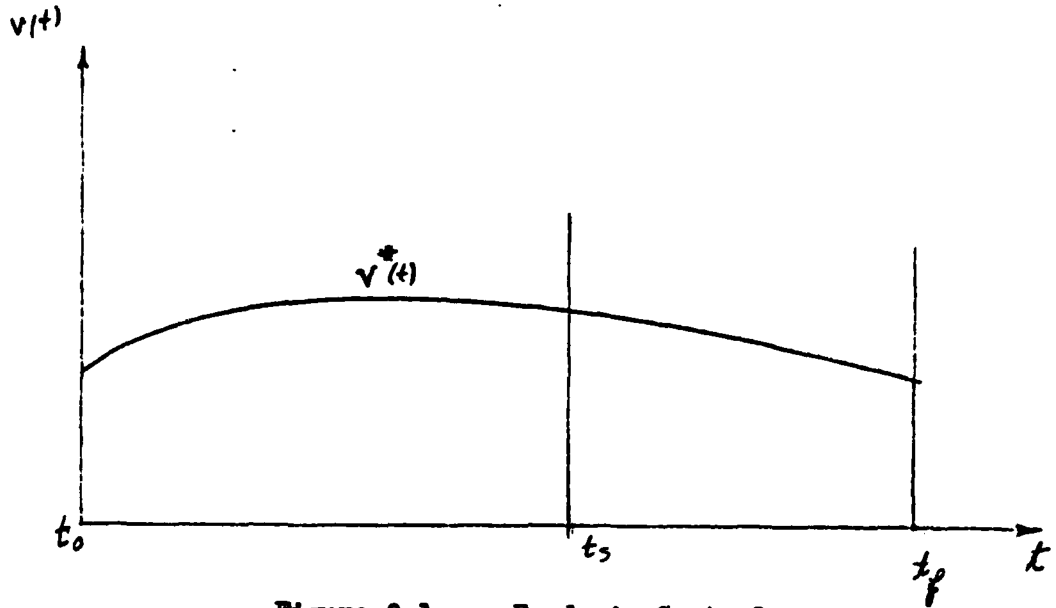

Figure 3.1-a. Evader's Control

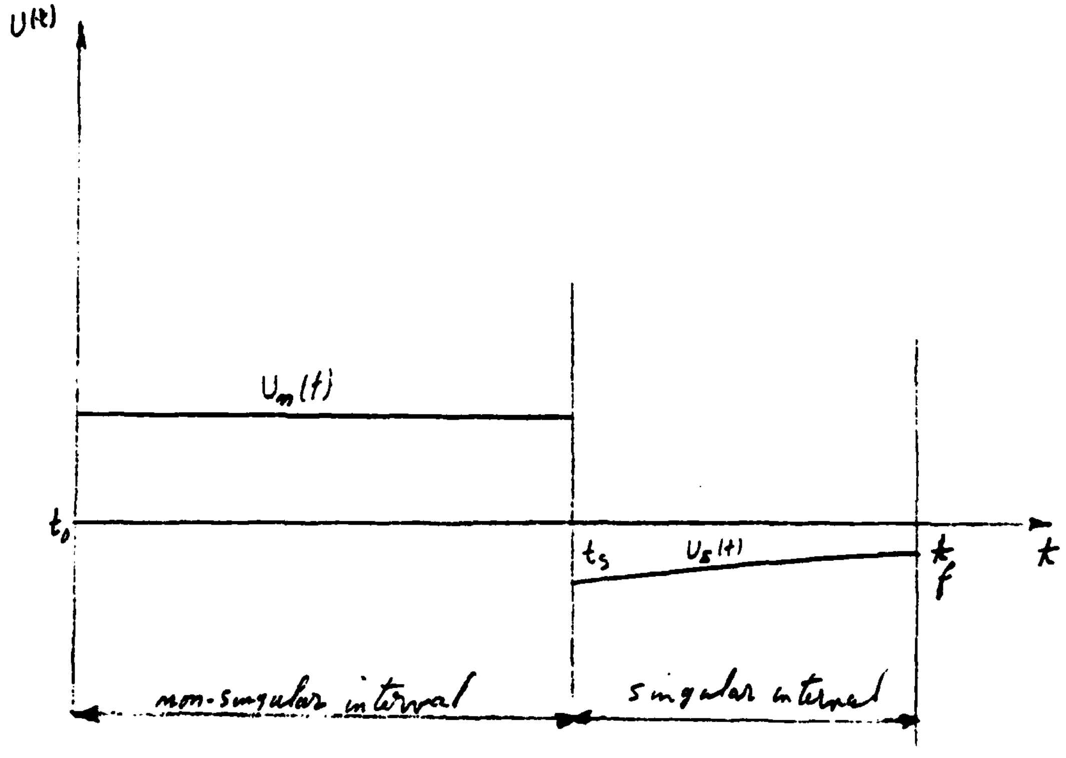

Figure 3.1-b. Pursuer's Control

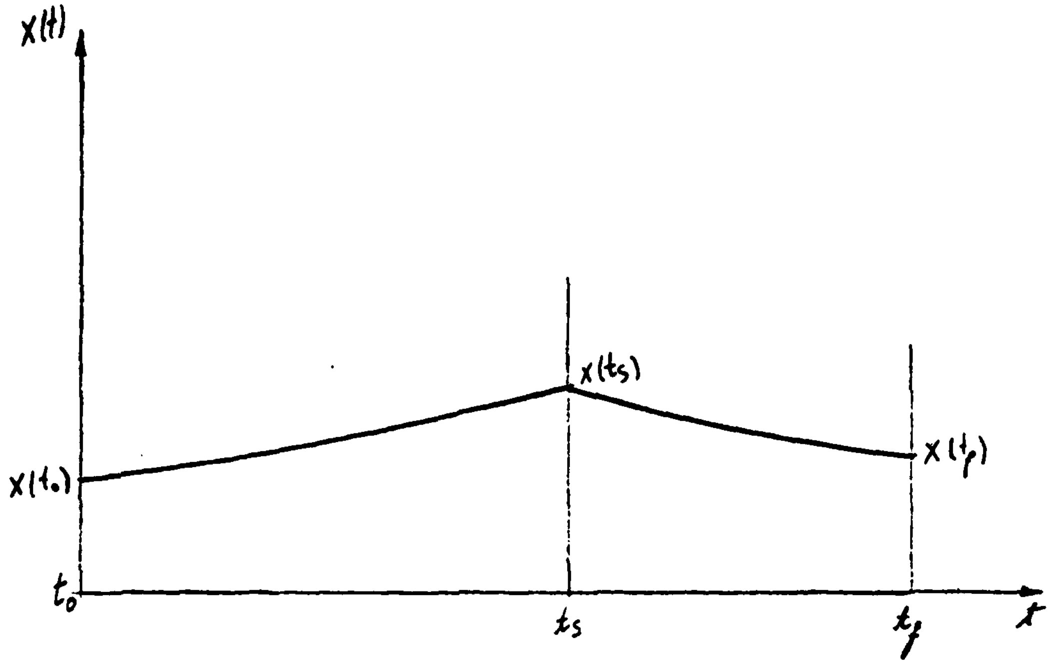

Figure 3.1-c. State Trajectory of the Game.

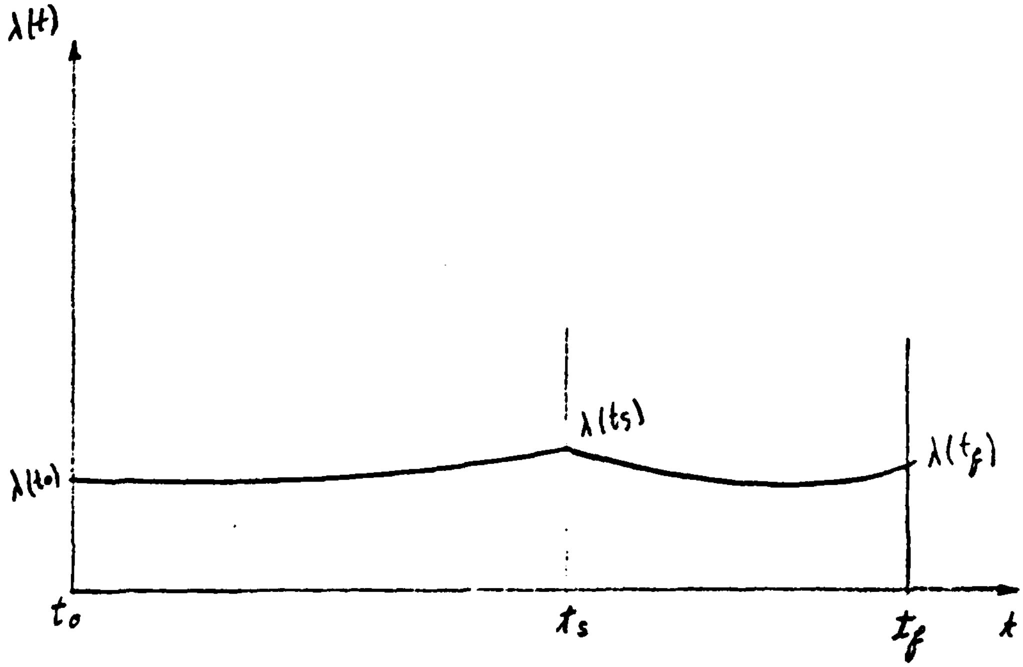

Figure 3.1-d. Costate Trajectory of the Game.

the singular arc including t_s^+. Continuity condition (3.1-1) and (3.1-2) together with (3.1-3) implies that (3.1-3) hold at t_s^- on the non-singular arc and provides n conditions at the junction point. So a set of multipoint boundary value problems is formed. Also we define a Terminal Error Vector Function TEF as

$$\text{TEF} = \lambda(t_f) - Sx(t_f) \tag{3.1-4}$$

which is a boundary condition. Since the time t_s is unknown we establish a procedure that iterates on t_s and drives euclidean norm $\text{N}(t_s) = \|\text{TEF}\|$ to zero, so that all the necessary and boundary conditions for optimality are satisfied. In order to derive the iterative relationship for the switching time t_s we purturb switching time around a nominal value t_s^i. So, the final states and costates are purturbed as much as $\Delta x(t_f)$ and $\Delta\lambda(t_f)$. Figure (3.2-a,b) shows the change of final state and costate due to the change of switching time. If

$$\text{TEF}^i = \lambda^i(t_f) - Sx^i(t_f) \tag{3.1-5}$$

where $x^i(t_f)$ and $\lambda^i(t_f)$ are the final state and costate obtained by the choice of t_s^i. So

$$\text{TEF} = \text{TEF}^i + \Delta\text{TEF} = \lambda^i(t_f) + \Delta\lambda(t_f) - Sx^i(t_f) - S\Delta x\ (t_f) \tag{3.1-6}$$

We express $\Delta x(t_f)$ and $\Delta\lambda(t_f)$ as a function of Δt_s where

$$\Delta t_s = t_s - t_s^i \tag{3.1-7}$$

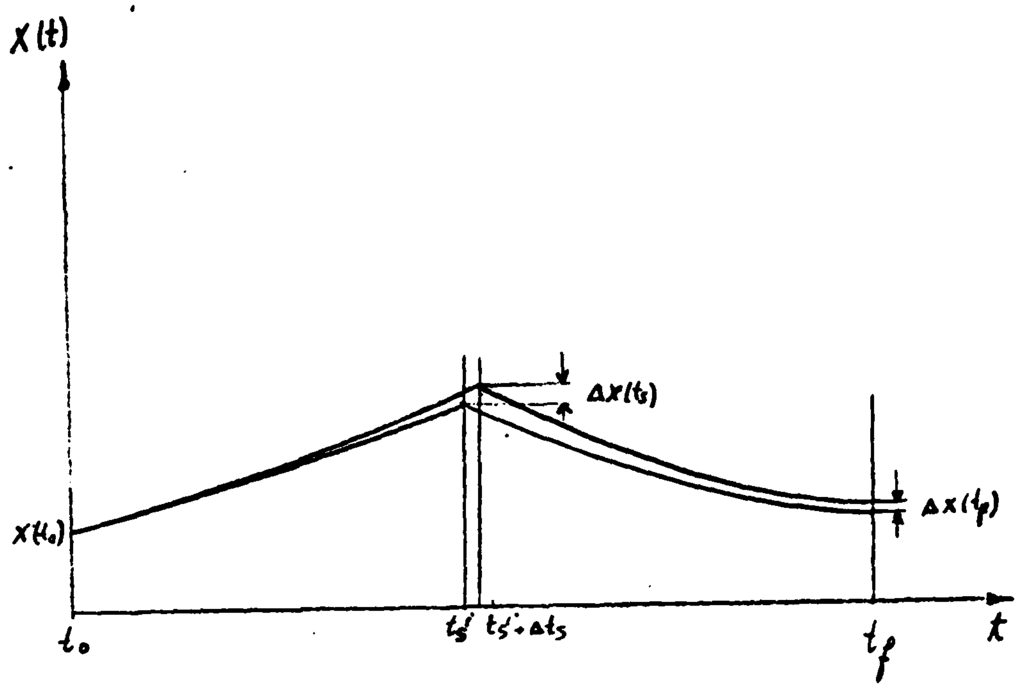

Figure 3.2-a. State Trajectory and Nieghboring Comparison Curve

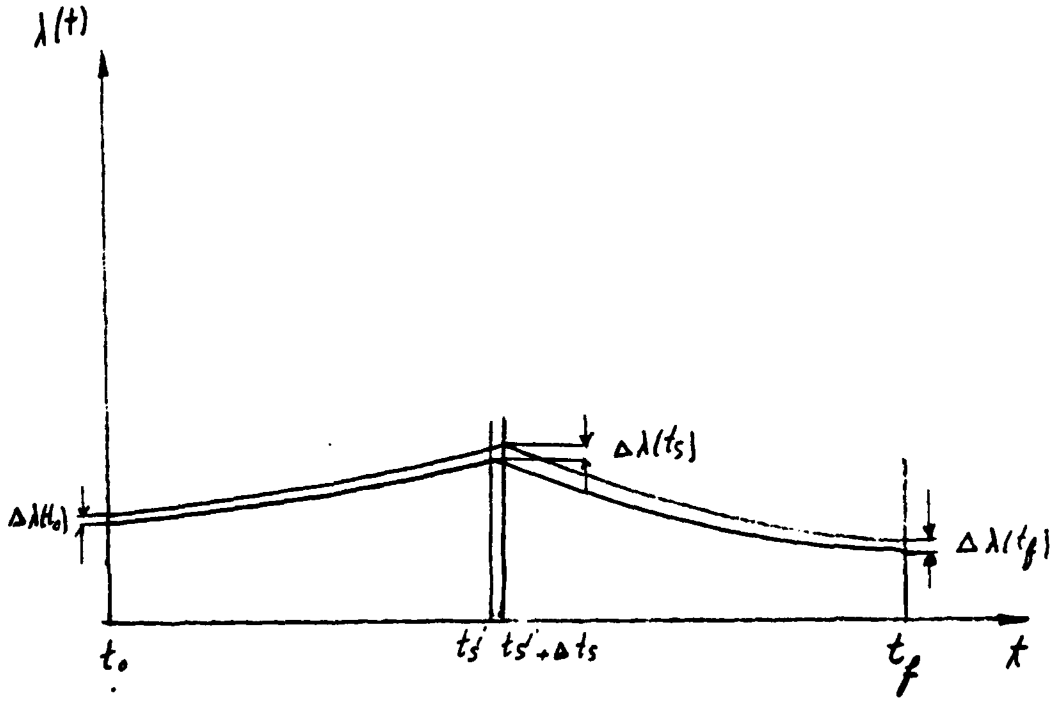

Figure 3.2-b. Costate Trajectory and Neighboring Comparison Curve.

For this purpose we solve (2.6-44) and (2.6-45) for the interval $t \in [t_0, t_s]$

$$\begin{bmatrix} x(t_s) \\ \lambda(t_s) \end{bmatrix} = [\Phi(t_s,t_0)] \begin{bmatrix} x(t_0) \\ \lambda(t_0) \end{bmatrix} + [\eta(t_s)] \tag{3.1-8}$$

$$[\dot{\Phi}(t_s,t_0)] = \begin{bmatrix} A & -GR^{-1}G^T \\ -Q & -A^T \end{bmatrix} [\Phi(t_s,t_0)] \tag{3.1-9}$$

$$\Phi(t_s,t_0) = I \tag{3.1-10}$$

and

$$\eta(t_s) = \int_{t_0}^{t_d} \Phi(t_s,\tau) \begin{bmatrix} \pm BK \\ 0 \end{bmatrix} d\tau \tag{3.1-11}$$

solving (2.6-46) and (2.6-47) we get

$$\begin{bmatrix} x(t_f) \\ \lambda(t_f) \end{bmatrix} = [\psi(t_f,t_s)] \begin{bmatrix} x(t_s) \\ \lambda(t_s) \end{bmatrix} \tag{3.1-12}$$

where

$$[\dot{\psi}(t_f,t_s)] = \begin{bmatrix} A+BM & BN-CR^{-1}C^T \\ -Q & -A^T \end{bmatrix} [\psi(t_f,t_s)] \tag{3.1-13}$$

$$\psi(t_f,t_f) = I \tag{3.1-14}$$

According to (3.1-7) the following relationships hold

$$
\left.\begin{aligned}
x(t_s) &= x(t_s^i) + \Delta x(t_s) \\
\lambda(t_s) &= \lambda(t_s^i) + \Delta\lambda(t_s) \\
\eta(t_s) &= \eta(t_s^i) + \Delta\eta(t_s) \\
\Phi(t_s,t_0) &= \Phi(t_s^i,t_0) + \Delta\Phi(t_s) \\
\psi(t_f,t_s) &= \psi(t_f,t_s^i) + \Delta\psi(t_s)
\end{aligned}\right\} \qquad (3.1.15)
$$

where $\Delta\eta(t_s)$, $\Delta\Phi(t_s)$ and $\Delta\psi(t_s)$ can be expressed as a function of Δt_s.

From (3.1-8) and (3.1-12) and (3.1-15) the following relationship can be obtained.

$$
\begin{bmatrix} \Delta x(t_s) \\ \Delta\lambda(t_s) \end{bmatrix} = [\Phi(t_s^i,t_0)] \begin{bmatrix} \Delta x(t_x) \\ \Delta\lambda(t_0) \end{bmatrix} + \Delta\Phi(t_s) \begin{bmatrix} x(t_0) \\ \lambda(t_0) \end{bmatrix} + \Delta\eta(t_s) \qquad (3.1\text{-}17)
$$

and

$$
\begin{bmatrix} \Delta x(t_f) \\ \Delta\lambda(t_f) \end{bmatrix} = [\psi(t_f,t_s^i)] \begin{bmatrix} \Delta x(t_s) \\ \Delta\lambda(t_s) \end{bmatrix} + [\Delta\psi(t_s)] \begin{bmatrix} x(t_s^i) \\ \lambda(t_s^i) \end{bmatrix} \qquad (3.1\text{-}18)
$$

Expressing $\Delta\Phi(t_s)$, $\Delta\psi(t_s)$, $\Delta\eta(t_s)$ and $\Delta\lambda(t_0)$ as functions of Δt_s to the first order, we can derive $\Delta x(t_f)$ and $\Delta\lambda(t_f)$ in terms of Δt_s. By substitution from (3.1-18) and (3.1-19) in (3.1-6) we can obtain TEF and $\frac{\partial \mathrm{TEF}}{\partial t_s}$ at each t_s. Therefore we can obtain several iterative relationships to derive TEF to zero.

The switching time can be obtained as

$$t_s^{i+1} = t_s^i + [\frac{\partial}{\partial t_s} \eta(t_s^i)]^{-1} \eta(t_s^i) \qquad (3.1\text{-}17)$$

Or in the least square sense

$$t_s^{i+1} = t_s^i + \frac{\partial TEF^{\dagger}}{\partial t_s}\bigg|_i TEF_i \qquad (3.1\text{-}18)$$

where

$$\frac{\partial TEF^{\dagger}}{\partial t_s} = \{[\frac{\partial TEF}{\partial t_s}]^T [\frac{\partial TEF}{\partial t_s}]\}^{-1} [\frac{\partial TEF}{\partial t_s}] \qquad (3.1\text{-}19)$$

The details of the derivation are given in Appnedix B. To see how this technique is accomplished, we outline the algorithm as it would be executed if a digital computer were used.

Algorithm

Step 1. Select a nominal switching time t_s^i, $t_0 < t_s^i < t_f$ assuming this choice is sufficiently close to the optimal switching time t_s^*. Let the iteration index i be zero.

Step 2. Integrate the state and costate equations (2.6-44) and (2.6-45) from t_0 to t_s as a set of TPBVP with boundary conditions (2.6-3) and (3.1-3) and store $x(t_s^i)$ and $\lambda(t_s^i)$.

Step 3. Integrate the state of costate equations (2.6-46) and (2.6-47) from t_s^i to t_f as a set of initial value problems with initial condition

$x(t_s^i)$ and $\lambda(t_s^i)$.

Step 4. If

$$\|TEF(t_s^i)\| \le \gamma \tag{3.1-20}$$

where γ is preselected positive constant the problem is solved, otherwise compute t_s^{i+1} from (3.1-17) and repeat the procedure from Step 2 to Step 4.

3.2 Extension of the Numerical Algorithm to a Class of Non-Linear Games

Let the set of state and costate equations obtained from the application of the necessary condition for the saddle point be

$$\begin{cases} \dot{x}(t) = f(x,\lambda,t) & (3.2\text{-}1) \\ \dot{\lambda}(t) = g(x,\lambda,t) & (3.2\text{-}2) \end{cases}$$

for non-singular interval $[t_0,t_s^i]$ and for singular interval $[t_s^i,t_f]$ be

$$\begin{cases} \dot{x} = \tilde{f}(x,\lambda,t) & (3.2\text{-}3) \\ \dot{\lambda} = \tilde{g}(x,\lambda,t) & (3.2\text{-}4) \end{cases}$$

$$x(t_0) = x_0 \quad \text{given} \tag{3.2-5}$$

$$\lambda(t_f) = \frac{\partial h(x(t_f))}{\partial x} \tag{3.2-6}$$

Define

$$\mathrm{TEF} = \lambda(t_f) - \frac{\partial h(x(t_f))}{\partial x} \tag{3.2-7}$$

For an iterative relationship we try to obtain TEF and $\frac{\partial \mathrm{TEF}}{\partial t_s}$ for each value of switching time t_s^i.

From Figures (3.3-a,b) we get

$$\begin{cases} \delta x(t_s^-) \simeq \Delta x(t_s) - \dot{x}(t_s^-)\Delta t_s & (3.2\text{-}8) \\ \delta\lambda(t_s^-) \simeq \Delta\lambda(t_s) - \dot{\lambda}(t_s^-)\Delta t_s & (3.2\text{-}9) \end{cases}$$

$$\begin{cases} \delta x(t_s^+) \simeq \Delta x(t_s) + \dot{x}(t_s^+)\Delta t_s & (3.2\text{-}10) \\ \delta\lambda(t_s^+) \simeq \Delta\lambda(t_s) + \dot{\lambda}(t_s^+)\Delta t_s & (3.2\text{-}11) \end{cases}$$

Linearizing (3.2-1) - (3.2-4) around a trajectory obtained due to the choice of t_s^i we will have

$$\begin{cases} \delta\dot{x} = \frac{\partial f}{\partial x}\,\delta x + \frac{\partial f}{\partial \lambda}\,\delta\lambda & (3.2\text{-}12) \\ \delta\dot{\lambda} = \frac{\partial g}{\partial x}\,\delta x + \frac{\partial g}{\partial \lambda}\,\delta\lambda & (3.2\text{-}13) \end{cases}$$

$$\begin{cases} \delta\dot{x} = \frac{\partial \tilde{f}}{\partial x}\,\delta x + \frac{\partial \tilde{f}}{\partial \lambda}\,\delta\lambda & (3.2\text{-}14) \\ \delta\dot{\lambda} = \frac{\partial \tilde{g}}{\partial x}\,\delta x + \frac{\partial \tilde{g}}{\partial \lambda}\,\delta\lambda & (3.2\text{-}15) \end{cases}$$

$$\delta x(t_0) = 0 \tag{3.2-16}$$

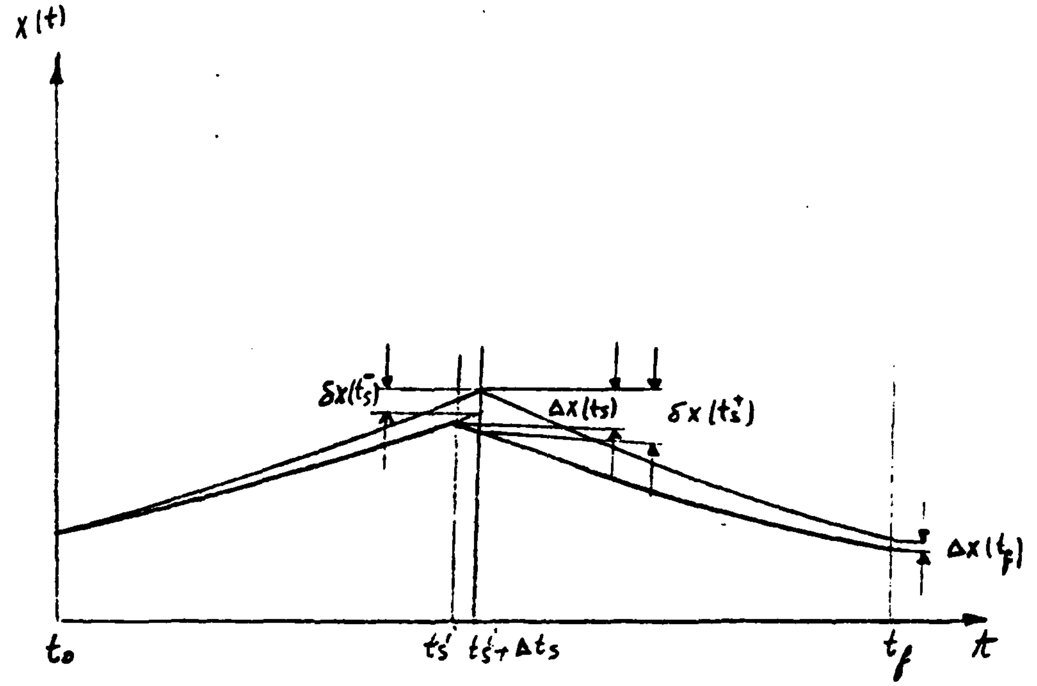

Figure 3.3-a. State Trajectory and Neighboring Comparison Curve.

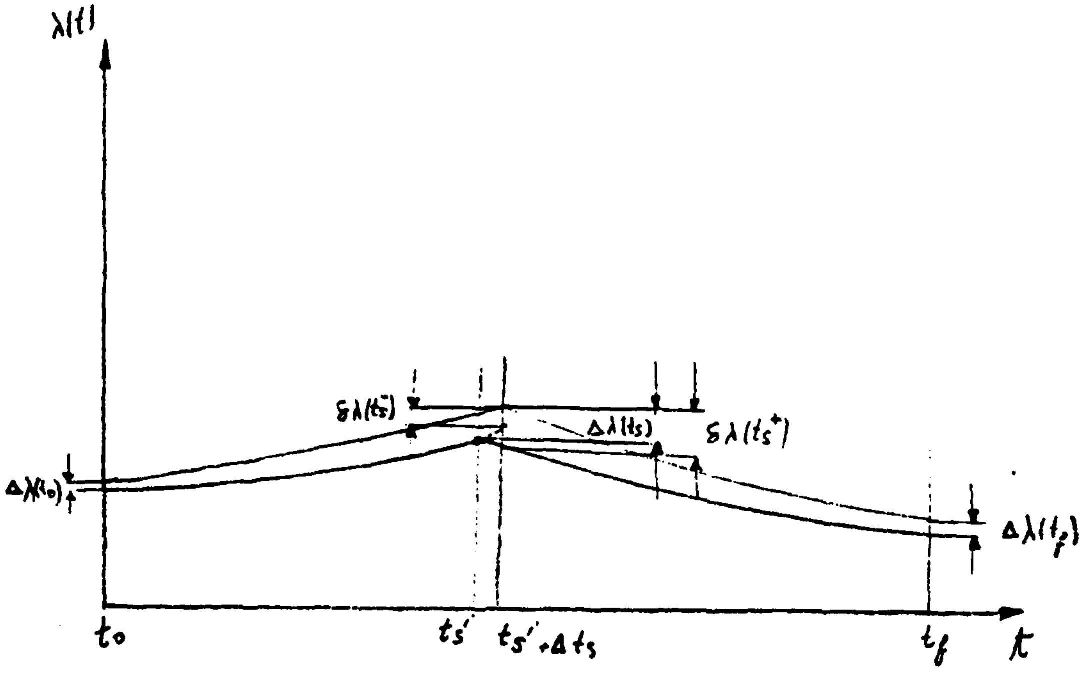

Figure 3.3-b. Costate Trajectory and Neighboring Comparison Curve

If the switching time is perturbed as much as $\Delta t_s = t_s - t_s^i$, then,

$$TEF = TEF^i + \Delta TEF \tag{3.2-17}$$

where ΔTEF is the increment of TEF due to the increment Δt_s. We can express the following relationship

$$TEF = \underline{\lambda}^i(t_f) + \delta\lambda(t_f) - \frac{\partial h}{\partial x}\bigg|_i - \frac{\partial^2 h}{\partial x^2}\,\delta x(t_f) + \text{H.O.T.} \tag{3.2-18}$$

By solving (3.2-12) - (3.2-14) and substituting in (3.2-18) for $\delta x(t_f)$ and $\delta\lambda(t_f)$ in terms of Δt_s we find ΔTEF as a function of Δt_s so we will be able to find $\frac{\partial TEF}{\partial t_s}$.

The iterative formula for the switching time is the same as (3.1-17) or (3.1-18) and the derivation is given in Appendix B.

The procedure is carried out on a digital computer similar to the linear case, the only difference is the computation of $\frac{\partial TEF}{\partial t_s}$, which is done through linearization according to the Appendix B.

In the case that there are two switching points the sequence of singular and non-singular controls are estimated. For example, assume the game starts with a non-singular arc and at time t_{s_1} switches to a singular arc and at time t_{s_2} switches to a non-singular arc and the game terminates on the non-singular arc. t_{s_1} and t_{s_2} are initially guessed and it is tried to drive TEF to zero with respect to t_{s_1} and t_{s_2} which are independent of each other.

3.3 Second Approach

In some high dimensional problems the number of junction conditions may be $m < n$. In such a case the first approach should be modified and Terminal Error Function should be driven to zero by iterating on the switching times and $n-m$ initial costates, i.e., $\lambda_j(t_0)$, $j = 1,2,\ldots,n-m$. For the class of problems with linear state equations TEF is expressed as a function of initial costates $\lambda(t_0)$ and switching time. In this approach for a computed switching time t_s^i, initial costates $\lambda^i(t_0)$ are also computed to minimize $h_i(\lambda(t_0),t_s^i) = \|TEF\|_i$ at the i^{th} iteration, and then at each t_s the gradient of $\mathrm{Min}_{\lambda(t_0)} \|TEF\|$ with respect to t_s is obtained. In order to satisfy the junction conditions, at each step of minimization m constraints on initial costates is considered. By finding the minimum value of h and its gradient with respect to t_s at each t_s^i we will be able to find a new t_s^i which yields a smaller h, i.e., (by Newton method). This procedure is continued until $\|TEF\| = 0$, in which case the problem is solved.

The value of the error at final time for each iteration as a function of switching time and initial costates is obtained as

$$h(t_s^i,\lambda(t_0)) = \bar{P}(t_s^i) + \bar{Q}(t_s^i)\lambda^i(t_0) + \lambda^{i^T}(t_0)\bar{R}(t_s^i)\lambda^i(t_0) \quad (3.3\text{-}1)$$

where $\bar{P}(t_s^i)$, $\bar{Q}(t_s^i)$ and $\bar{R}(t_s^i)$ are functions of switching time given in Appendix C. Also m conditions at the junction point yields m constraint relationships on the components of $\lambda(t_0)$ as

$$W_2(t_s^i)\lambda(t_0) = T(t_s^i) - W_1(t_s^i)x(t_0) \quad (3.3\text{-}2)$$

where $W_1(t_s^i)$ and $W_2(t_s^i)$ are $m \times n$ matrices and $T(t_s^i)$ is an m vector specified in Appendix C.

The following iterative relationship may be used to find the optimal switching time

$$t_s^{i+1} = t_s^i + \left[\frac{\partial}{\partial t_s} \eta\right]^{-1}_{t_{s_i}} \left[\underset{\lambda(t_0)}{\text{Min}} \ \eta\right]_{t_s^i} \tag{3.3-3}$$

The derivation of $\frac{\partial}{\partial t_s} \eta$ is shown in Appendix C.

The following algorithm is the outline of the steps required to carry out this approach.

Algorithm

Step 1. Select a nominal switching time $t_0 < t_s^i < t_f$ sufficiently close to the optimal switching time set $i = 0$.

Step 2. Compute $\min_{\lambda(t_0)} \eta(\lambda(t_0), t_s^i)$ from (3.3-1) and (3.3-2). Matrices $\overline{R}(t_s^i)$, $\overline{Q}(t_s^i)$, $\overline{P}(t_s^i)$, $W(t_s^i)$, $W_2(t_s^i)$ and $T(t_s^i)$ should be computed a priori.

Step 3. If

$$\min_{\lambda(t_0)} \eta(\lambda(t_0), t_s^i) \le \gamma$$

where γ is a preselected positive scalar number, the problem is solved, otherwise go to Step 4.

Step 4. Compute $\frac{\partial}{\partial t_s} [\min_{\lambda(t_0)} \eta(t_s)]_{t_s^i}$, and find t_s^{i+1} from (3.3-3) go to Step 2, repeat the procedure.

3.4 Features of the Numerical Algorithms

The important features of the algorithms are as follows:

1. Initial Guess. To begin the procedure a guess for initial switching time should be made. This guess is usually based on the physical nature of the problem.

2. Storage Requirement. For linear problems only initial state and terminal conditions and the values of states and costates at junction points should be stored. For non-linear cases each trajectory obtained at each iteration is stored as a linearizing trajectory for the next iteration.

3. Convergence. Since the method has the characteristic of the Newton method, if t_s^i is sufficiently close to the optimal t_s the method will generally converge quite rapidly. However, if the initial guess is very poor, the method may not converge at all.

4. Computations Required. In the first approach at each iteration we need to invert an $n \times n$ matrix and solve a set of TPBVP and in the second approach we have to solve a non-linear programming problem at each iteration.

5. Stopping Criterion. In problems with fixed final states the procedure is exactly the same as free final state. In this case the terminal error function will be defined as

$$TEF = x(t_f) - x_f$$

where x_f is a given vector for final states and the procedure terminates if $\|TEF\| \leq \gamma$.

Remark. The second approach can directly handle problems in which the sequence of control is bang bang.

3.5 Numerical Example.

Consider the class of generalized prusuit-evasion problems (2.6-1) - (2.6-4) where $S = 0$, $Q = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, $R = 100$, $A = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$, $B = \begin{bmatrix}0\\1\end{bmatrix}$, $C = \begin{bmatrix}0\\1\end{bmatrix}$, $t_0 = 0$, $t_f = 3.17$, $x(0) = \begin{bmatrix}0.59845\\0.60706\end{bmatrix}$ and $K = 1$. Assuming both players have perfect information and measurements of the output and parameters of the system and

$$y(t) = x(t) \tag{3.5-1}$$

for this problem, Condition

$$\int_0^{3.17} t\left(\int_0^t \|e^{A(t-\tau)}\, CR^{-1/2}\|_1^2 d\tau\right)\|Q\|_1 dt = .21340 < 1 \tag{3.5-2}$$

and all other required conditions for the existence and uniqueness of the saddle point are satisfied. For given matrices and $t_f = 3.17$ any $R \geq 21.5$ satisfies (3.5-2).

Table (3.1) shows the solution to the matrix Ricatti equation (2.6-52) which is finite in $0 \leq t \leq 3.17$ is an alternative to (3.5-2) for the uniqueness and the existence of the saddle point. For the necessary conditions. Define Hamiltonian

$$H = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - \frac{100}{2}v^2 + \lambda_1 x_2 + \lambda_2 u + \lambda_2 v \tag{3.5-3}$$

$t_f - t$	P_1	P_2	P_3	P_4
0.0	0.0	0.0	0.0	0.0
0.12500	0.12500	0.00781	0.00781	0.12566
0.25000	0.25000	0.03125	0.03125	0.25526
0.37500	0.37500	0.07034	0.07034	0.39277
0.50000	0.50002	0.12508	0.12508	0.54214
0.62500	0.62505	0.19552	0.19552	0.70738
0.75000	0.75012	0.28171	0.28171	0.89251
1.00000	1.00050	0.50162	0.50162	1.33879
1.25000	1.25153	0.78571	0.78571	1.91451
1.50000	1.50384	1.13558	1.13558	2.65503
1.75000	1.75836	1.55378	1.55378	3.59879
2.00000	2.01644	2.04435	2.04435	4.78900
2.25000	2.27998	2.61340	2.61340	6.27625
2.50000	2.55159	3.26997	3.26997	8.12232
2.75000	2.83463	4.02727	4.02727	10.40588
3.00000	3.13460	4.90463	4.90463	13.23125
3.25000	3.45782	5.93045	5.93045	16.74236
3.50000	3.81443	7.14721	7.14721	21.14589
3.75000	4.21928	8.62019	8.62019	26.75177
3.87500	4.44659	9.48308	9.48308	30.14783
4.00000	4.69565	10.45359	10.45359	34.04814
4.12500	4.97182	11.55611	11.55611	38.56596
4.25000	5.28226	12.82333	12.82333	43.85283
4.37500	5.63671	14.29998	14.29998	50.11600
4.50000	6.04888	16.04901	16.04901	57.64680
4.62500	6.53859	18.16179	18.16179	66.86786
4.68750	6.82130	19.39503	19.39503	72.29935
4.75000	7.13570	20.77602	20.77602	78.41615
4.81250	7.48828	22.33467	22.33467	85.35667
4.87500	7.88741	24.10966	24.10966	93.29961
4.93750	8.34407	26.15175	26.15175	102.47968
5.00000	8.87297	28.52893	28.52893	113.21143
5.06250	9.49432	31.33450	31.33450	125.92627
5.12500	10.23654	34.70000	34.70000	141.23239
5.15625	10.66522	36.64926	36.64926	150.11884
5.18750	11.14106	38.81682	38.81682	160.01514
5.21875	11.67261	41.24223	41.24223	171.10440
5.25000	12.27066	43.97531	43.97531	183.61688
5.28125	12.94892	47.07959	47.07959	197.84625
5.31250	13.72521	50.63733	50.63733	214.17308
5.34375	14.62299	54.75710	54.75710	233.09937
5.37500	15.67390	59.58513	59.58513	255.30130
5.39063	16.26978	62.32489	62.32489	267.90869
5.40625	16.92152	65.32309	65.32309	281.71118
5.42188	17.63751	68.61841	68.61841	296.88770
5.43750	18.42786	72.25758	72.25758	313.65430
5.45313	19.30490	76.29770	76.29770	332.27539
5.46875	20.28389	80.80933	80.80933	353.07666
5.48438	21.38388	85.88049	85.88049	376.46558
5.50000	22.62898	91.62273	91.62273	402.95752

Table 3.1. Matrix Ricatti Solution for $R = 100$

$$\frac{\partial H}{\partial v} = -100v + \lambda_2 = 0 \tag{3.5-4}$$

$$v = \frac{\lambda_2}{100} \tag{3.5-5}$$

Costate equations are

$$\begin{cases} \dot{\lambda}_1 = -x_1 & (3.5\text{-}6) \\ \dot{\lambda}_2 = -\lambda_1 - x_2 & (3.5\text{-}7) \end{cases}$$

$$u_n = \begin{cases} +1 \quad \text{if} \quad \lambda_2 < 0 & (3.5\text{-}8) \\ -1 \quad \text{if} \quad \lambda_2 > 0 & (3.5\text{-}9) \end{cases}$$

$$u_s = x_1 \quad \text{if} \quad \lambda_2 = 0$$

along the singular arc the stregnthened GLC condition

$$\frac{\partial}{\partial u} \frac{d^2}{dt^2} \frac{\partial H}{\partial u} = -1 < 0 \tag{3.5-10}$$

is satisfied, and transversality conditions are

$$\lambda_1(3.17) = 0 \tag{3.5-11}$$

$$\lambda_2(3.17) = 0 \tag{3.5-12}$$

In this example a sequence of control $u_n = -1$ for $0 \leq t \leq t_s$ and $u_s = -x_1$ for $t_s < t \leq 3.17$ with one switching from non-singular control to singular control satisfied all the necessary and transversality conditions for the optimality. To integrate the set of state and costate equations (2.6-44) - (2.6-47) with boundary conditions (2.6-3), (2.6-29), (2.6-37) and (2.6-38), by the first approach we used a computer program in SSP subroutine LBVP for TPBVP and subroutine HPCL for initial value problems. For the initial guess $t_s = .8$ after six iterations the problem stopped with

$$\eta = 6 \times 10^{-6} < 10^{-5} \tag{3.5-13}$$

the optimal switching time,

$$t_s = 1.195046 \simeq 1.2 \tag{3.5-14}$$

and the performance index,

$$J = .60243 \tag{3.5-15}$$

Table 3.2 shows iterative and computer results for this problem. Tables 3.3-a,b) show computer printout of the state and costate trajectories on singular and non-singular intervals. Figures 3.4-3.7 illustrate the behavior of the controls and trajectories. The computational time at each iteration with IBM 370 was approximately .54 second.

Iteration No. i	Switching Time t_s^i	$\frac{\partial}{\partial t_s} h$	Terminal Error h
1	.8	56.058021	9.137982
2	.973001	23.236232	2.370764
3	1.085023	9.150887	.515614
4	1.151342	3.262189	.987125
5	1.188061	.667985	.005082
6	1.195046		6×10^{-6}

Table 3.2. Computation of Switching Time t_s

Saddle Point Solution

Time	x_1	x_2	λ_1	λ_2
-0.00000	0.59847	0.60706	1.45739	1.24175
0.10000	0.65424	0.50820	1.39467	1.04333
0.20000	0.70011	0.40915	1.32687	0.86135
0.30000	0.73626	0.30993	1.25498	0.69627
0.40000	0.76209	0.21055	1.17999	0.54847
0.50000	0.77817	0.11103	1.10290	0.41824
0.60000	0.78429	0.01140	1.02469	0.30573
0.70000	0.78045	-0.08835	0.94637	0.21103
0.80000	0.76662	-0.18819	0.86893	0.13410
0.90000	0.74281	-0.28807	0.79338	0.07482
0.95000	0.72716	-0.33804	0.75662	0.05172
1.00000	0.70901	-0.38802	0.72071	0.03294
1.05000	0.68836	-0.43801	0.68576	0.01844
1.10000	0.66527	-0.48800	0.65191	0.00815
1.15000	0.63955	-0.53800	0.61928	0.00203
1.20000	0.61141	-0.58800	0.58800	0.0

Table 3.3-a. Numerical Results for Non-Singular Subarc

Time	x_1	x_2	λ_1	λ_2
1.20000	0.61141	-0.58800	0.58800	0.0
1.25000	0.58276	-0.55815	0.55815	0.0
1.30000	0.55557	-0.52970	0.52970	0.0
1.35000	0.52977	-0.50257	0.50257	0.0
1.40000	0.50529	-0.47670	0.47670	0.0
1.45000	0.48208	-0.45202	0.45202	0.0
1.50000	0.46007	-0.42847	0.42847	0.0
1.60000	0.41946	-0.38453	0.38453	0.0
1.70000	0.38304	-0.34444	0.34444	0.0
1.80000	0.35045	-0.30780	0.30780	0.0
1.90000	0.32138	-0.27423	0.27423	0.0
2.00000	0.29551	-0.24341	0.24341	0.0
2.10000	0.27261	-0.21503	0.21503	0.0
2.20000	0.25244	-0.18880	0.18880	0.0
2.30000	0.23479	-0.16446	0.16446	0.0
2.40000	0.21949	-0.14176	0.14176	0.0
2.50000	0.20639	-0.12049	0.12049	0.0
2.60000	0.19535	-0.10042	0.10042	0.0
2.70000	0.18627	-0.08135	0.08135	0.0
2.80000	0.17905	-0.06310	0.06310	0.0
2.90000	0.17363	-0.04548	0.04548	0.0
3.00000	0.16994	-0.02832	0.02832	0.0
3.09999	0.16795	-0.01144	0.01144	0.0
3.19999	0.16765	0.00533	-0.00533	0.0

Table 3.3-b. Numerical Results for Singular Subarc.

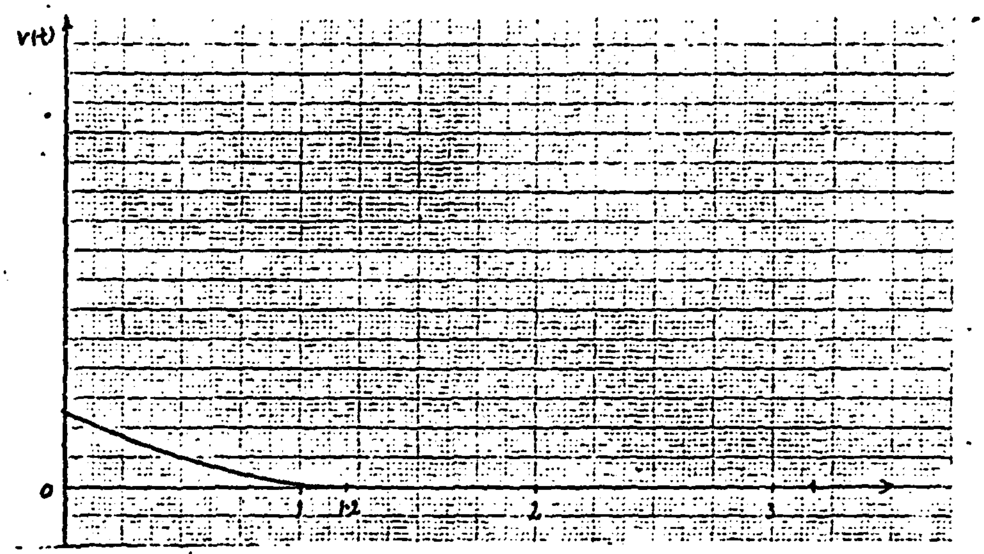

Figure 3.4. Evader's Saddle Point Strategy

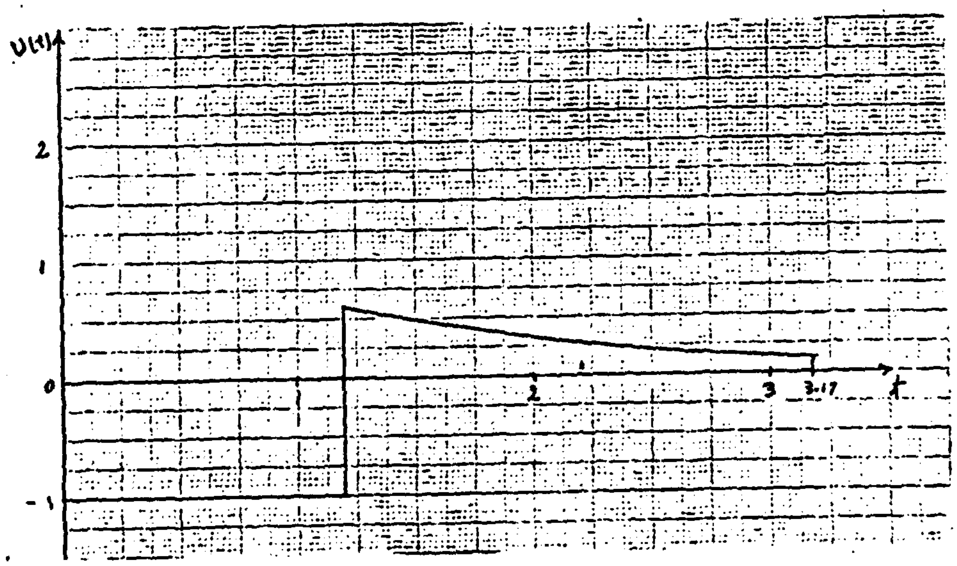

Figure 3.5. Pursuer Saddle Point Strategy

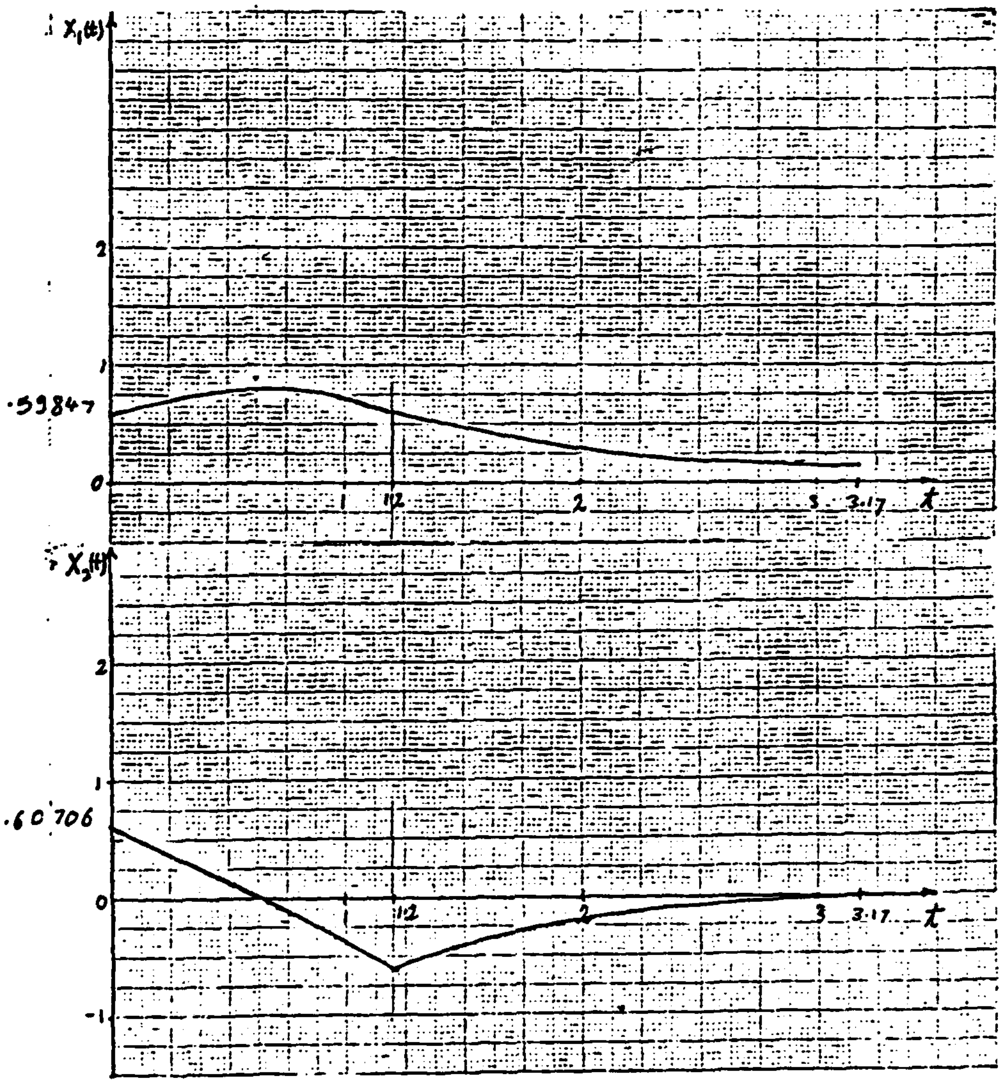

Figure 3.6. Sadle Point State Trajectories of the System

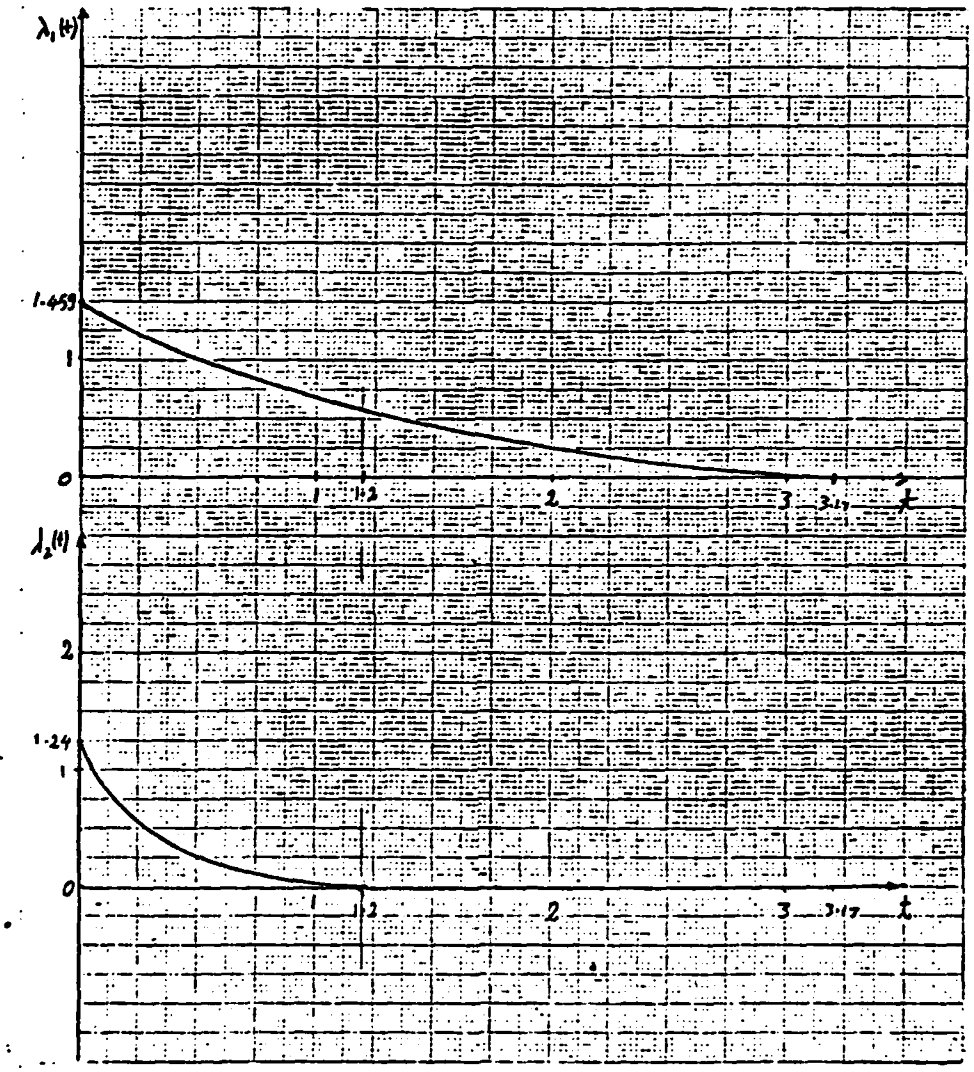

Figure 3.7. Saddle Point Costate Trajectories

CHAPTER 4

CLOSED-LOOP STRATEGY DEPENDENT SOLUTION AND APPLICATION OF INVERSE SYSTEMS IN SINGULAR PROBLEMS

As was mentioned before, in differential game problems closed-loop solutions are more desirable than open loop solutions.

In a closed loop solution each player's strategy is based upon information about the current state of his opponent and assumes that the opponent plays optimally. In the case where one player plays non-optimally, the opponent might be able to perform even better than the closed loop solution if he can determine the nonoptimal strategy of the other player. In this case the known strategy of the opponent can be assumed as an external input to the system and the game problem is converted to a one-sided optimal control problem. It is obvious that the performance that can be achieved in such a way would be better than the performance achievable either with open loop or closed loop strategies. The following static example will illustrate how the deviation of one player in a zero-sum game from the saddle point solution may affect the payoff of the game and how the opponent can achieve different performances.

4.1 Example

Consider the zero sum game

$$J(u,v) = u^2 - v^2 + uv \tag{4.1-1}$$

where u is minimizing J and v is maximizing it.

This problem has a saddle point solution defined by the u^* and v^* satisfying

$$\frac{\partial J}{\partial u} = 0, \qquad \frac{\partial J}{\partial v} = 0 \tag{4.1-2}$$

$$\frac{\partial^2 J}{\partial u^2} = 2 > 0, \quad \frac{\partial^2 J}{\partial v^2} = -2 < 0 \tag{4.1-3}$$

The saddle point solution is

$$u^* = 0, \qquad v^* = 0 \tag{4.1-4}$$

and the value of the game is

$$J(u^*,v^*) = 0 \tag{4.1-5}$$

Now, assume v chooses a strategy other than the saddle point $v^* = 0$, e.g. $\bar{v} = 1$ and u follows the same saddle point strategy $u^* = 0$. Then the value of the game will be

$$J(u^*,\bar{v}) = -1 \tag{4.1-6}$$

We note that u has achieved a better performance than the saddle point case. But u can achieve even a better performance than (4.1-6) based on the information that $\bar{v} = 1$. In this case it is enough to minimize (4.1-7)

$$J(u,\bar{v}) = u^2 + u - 1 \tag{4.1-7}$$

Thus

$$\frac{\partial J(u,\bar{v})}{\partial u} = 2u + 1 = 0 \tag{4.1-8}$$

so that

$$\hat{u} = -\frac{1}{2} \tag{4.1-9}$$

is optimum and

$$J(\hat{u},\bar{v}) = -1.25 < -1 \tag{4.1-10}$$

Consequently u has performed better in comparison with the case that he used his saddle point strategy.

This example shows even a reasonable approximate knowledge of v's strategy, can help u to do better than his saddle point strategy.

For example if $\bar{v} = 1$ and u has an approximate estimate of $\bar{v}$, e.g., $\bar{v} = .9$, then, going through the steps before to determine the optimal u,

$$J(u,\bar{v}) = u^2 + .9u - .81 \tag{4.1-11}$$

$$\frac{\partial J(u,\bar{v})}{\partial u} = 2\tilde{u} + .9 \tag{4.1-12}$$

$$\tilde{u} = -.45 \tag{4.1-13}$$

$$J(\tilde{u},\tilde{v}) = -1.0125 < -1 \tag{4.1-14}$$

So, u is still better off following this control rather that his saddle point strategy.

<u>Remark</u>. In general, sometimes a player may not be able to achieve better than his saddle point. For example consider the case

$$J = u - v^2 \tag{4.1-15}$$

$$|u| \leq 1 \tag{4.1-16}$$

where u is minimizing and v is maximizing J. The saddle point solution is

$$u^* = -1 \qquad v^* = 0 \tag{4.1-17}$$

If v deviates from the saddle point strategy the best strategy for u is to play the saddle point strategy $u = -1$, for any deviation of v. This case may occur for dynamic problems in the case that control $u(t)$ is always on one boundary of his control region. But in cases of bang bang and totally and partially singular $u(t)$ deviations of $v(t)$ can effect switching time and some change in singular control arc and even it may change the whole sequence of singular and non-singular $u(t)$.

In pursuit-evasion game problems some methods have been suggested to determine the opponent's strategy. These approaches may be categorized as estimation techniques[47] or inverse system techniques[56],[48]. In the past singularities have been excluded from these problems. In this chapter we will utilize the concept of inverse systems (assuming its existence) to determine the opponent's control through a state or output measurement. Thus, together with the proposed techniques of Chapter 3, enables us to generate an approximate closed loop strategy dependent solution. In the next section we will discuss the concept of inverse systems, and their existence, and then we will show how by such a system to determine the strategy of the opponent.

4.2 Inverse System

4.1.a Basic Definition. Let $\mathcal{U}$ and $\mathcal{Y}$ be sets. A mapping $\mathcal{S}:\mathcal{U} \to \mathcal{Y}$ is said to be invertible if there exists a mapping $\hat{\mathcal{S}}:\mathcal{Y} \to \mathcal{U}$ such that $\hat{\mathcal{S}}\mathcal{S}$ and $\mathcal{S}\hat{\mathcal{S}}$ are identity mappings on the sets $\mathcal{U}$ and $\mathcal{Y}$ respectively. In this case $\hat{\mathcal{S}}$ is said to be an inverse of $\mathcal{S}$. Since such an inverse if it exists, is unique, we will denote the inverse of $\mathcal{S}$ by $\mathcal{S}^{-1}$.

The following definition is due to Zadeh and Dessoer in Reference (3).

4.1.b. Formal Definition. Let $\mathcal{S}$ and $\hat{\mathcal{S}}$ be characterized by input-output-state relations of the form

$$\mathcal{S} : y = I(x;u) \qquad x \in \Sigma_x \tag{4.2-1}$$

$$\hat{\mathcal{S}} : w = J(z;v) \qquad z \in \Sigma_z \tag{4.2-2}$$

where u and v are inputs to $\mathcal{S}$ and $\hat{\mathcal{S}}$ respectively, and y and w represent the corresponding outputs. x and z are corresponding states of each system (the output function space of $\mathcal{S}$ is assumed to be the input function space of $\hat{\mathcal{S}}$ and conversely). Then $\hat{\mathcal{S}}$ is called inverse to $\mathcal{S}$ or $\mathcal{S}$ is called inverse to $\hat{\mathcal{S}}$ if and only if to every state x of $\mathcal{S}$ there exists a state z_x of $\hat{\mathcal{S}}$ such that

$$J(z_x;\, I(x;u)) = u \qquad \forall\, u \tag{4.2-3}$$

and conversely to every state z of $\hat{\mathcal{S}}$ there corresponds a state x_z of $\mathcal{S}$ such that

$$I(x_z;\, J(z,v)) = v \qquad \forall\, v \tag{4.2-4}$$

If $\mathcal{S}$ is inverse to $\hat{\mathcal{S}}$ then $\mathcal{S}$ is denoted by $\hat{\mathcal{S}}^{-1}$ and $\hat{\mathcal{S}}$ denoted by $\mathcal{S}^{-1}$. Correspondingly, the state z_x is denoted by x^{-1}. $\mathcal{S}$ will be said to be invertible if it has an inverse.

From this definition it follows that if (4.2-3) and (4.2-4) are satisfied with states x, z_x, and z, x_z respectively, then they are also satisfied with the states x_z, z and z_x, x.

For this study we will include theorems, without proof, and in the following we also include several pertinent definitions. For more detailed study of this subject references (3) and (10) are recommended.

<u>Theorem</u> 4.1. A mapping $\mathcal{S} : \mathcal{U} \to \mathcal{Y}$ is invertible if and only if it is one-to-one and onto.

Lemma 4.1. If a mapping $\mathcal{S} : \mathcal{U} \to \mathcal{Y}$ is invertible then $\mathcal{S}^{-1}$ is invertible and $(\mathcal{S}^{-1})^{-1} = \mathcal{S}$.

Definition 4.2. A mapping $\mathcal{S} : \mathcal{U} \to \mathcal{Y}$ is said to be

a) Pre-invertible or left invertible if a mapping $\hat{\mathcal{S}} : \mathcal{R}(\mathcal{S}) \to \mathcal{U}$ exists such that $\hat{\mathcal{S}}\mathcal{S} = I_{\mathcal{U}}$, where $\mathcal{R}(\mathcal{S})$ denotes the range of $\mathcal{S}$. In such a case $\hat{\mathcal{S}}$ is called a left or Post-invertible $\mathcal{S}$ and denoted by $\mathcal{S}_L^{-1}$.

b) Postinvertible or right invertible if a mapping $\bar{\mathcal{S}} : \mathcal{Y} \to \mathcal{U}$ exists such that $\mathcal{S}\bar{\mathcal{S}} = I_{\mathcal{Y}}$. In such a case $\bar{\mathcal{S}}$ is called a right or post-inverse of $\mathcal{S}$ and denoted by $\mathcal{S}_R^{-1}$.

Theorem 4.2. If $\mathcal{S} : \mathcal{U} \to \mathcal{Y}$ is both left and right invertible then $\mathcal{S}$ is invertible and $\mathcal{S}^{-1} = \mathcal{S}_L^{-1} = \mathcal{S}_R^{-1}$.

In the remainder of this chapter we will consider only linear dynamical systems indicated by the notation

$$\mathcal{S} \equiv (A,B,C,D) \tag{4.2-5}$$

This notation specifies the following set of state and output equations.

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{4.2-6}$$

$$y(t) = Cx(t) + Du(t) \tag{4.2-7}$$

where $x \in R^n$, $u \in R^m$, $y \in R^\ell$ are the state input and output vectors respectively. The matrices A, B, C and D are real, in general time varying, continuous and of compatible sizes. The initial conditions are assumed to be zero.

We now address ourselves to the issue of the existence of inverse systesm for (4.2-5), and the construction of such a system if indeed they do exist. Before we discuss conditions for the existence of such inverse systems and the algorithm to construct them, we briefly review the applications and the literature of this subject.

The concept of inverse systems has found applications in numerous problems of engineering. Information recovery, in coding theory is one of the areas in which inverse systems are used. A linear time invariant dynamical system can work as an encoder for a special type of code and the post inverse of the system is used as a decoder.

A post inverse system has been used in filtering and estimation theory in the presence of colored noise. It is used to whiten the colored noise which is easier to analyze.

In stochastic differential game problems the pursuer typically attempts to estimate the state of the evader's system, and then employs an inverse system to determine the evader's input.[58]

In the deterministic differential games inverse systems have been used to determine the opponent's strategies from the perfect measurement of the state of the system in non-singular problems.[48]

Other applications include decoupling of multivariable systems, network synthesis, network realization of passive impedences, and two point boundary value problems.

In the present work we apply this concept to develop a solution to deterministic linear differential game with singularities.

Most research which has been done in inverse systems is concerned with time invariant systems. Some authors have proposed algorithms which may be used for special classes of differential game problems. Our need is for some criteria for invertibility of the system and also an efficient algorithm for constructing the inverse system.

Patal[59] (1973) has obtained a sufficient condition for invertibility of a special class of time invariant systems in which $m \neq \ell$. This criterion simply tests the rank of the product of two matrices.

One of the most recent works for time invariant systems in the case of $\ell \neq m$ is the work of Sinwat and Fallside[50] (1976). They have proposed an algorithm which is based on the factorization of the transfer matrix of the system. The criterion for invertibility here, requires the formation of the transfer matrix and determination of its rank. Full rank of the transfer matrix is a necessary and sufficient condition for the applicability of their algorithm.

Silverman[51] introduced a finite sequential algorithm for time invariant systems and later on he extended the algorithm to the time varying systems. Also a sequential test of existence is incorporated in the algorithm.

4.2-1 Inversion of Linear Time Invariant Systems

With regard to the basic definition of the inverse system let $R[S]^{\ell \times m}$ be a ring of polynomial matrices. The linear time invariant system (4.2-6) and (4.2-7) whose transfer matrix function is $G(S) \in R[S]^{\ell \times m}$ and which is assumed to have full rank, is said to be left invertible if $\ell > m$ and there exists a system with transfer function matrix $G_L(S) \in R[S]^{\ell \times m}$ such that $G_L(S)G(S) = I_n$. G_L is called left inverse of $G(S)$. Similarly the system is said to be right invertible if $\ell < m$ and there exists a $G_R(S) \in R[S]^{\ell \times m}$ such that $G(S)G_R(S) = I_\ell$. $G_R(S)$ is called the right inverse of $G(S)$. When $\ell = m$ the right and left inverses are identical.

The following theorems which have been presented in references (59) and (50) give some conditions for invertibility of linear time invariant systems.

<u>Theorem</u> 4.3. A sufficient condition for invertibility of the linear time invariant systems (A,B,C) is that $\mathrm{rank}(CB) = \min(\ell,m)$.

<u>Theorem</u> 4.4. A necessary and sufficient condition for invertibility of the linear, time invariant system (A,B,C,D) is that $G(S)$ has a full rank.

4.2.2 <u>Inversion of Linear Time Varying Systems - Regular Systems</u>

Roughly speaking a regular system is a system in which $D(t)$ has a constant rank.

Given a set of regular linear time varying system

$$S: \begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) & (4.2\text{-}8) \\ \\ y(t) = C(t)x(t) + D(t)u(t) & (4.2\text{-}9) \end{cases}$$

for each given initial condition $x(t_0) = x_0$, Equations (4.2-8) and (4.2-9) define a mapping $H_{x_0}: \mathcal{U} \to \mathcal{Y}$. The following theorem applies to this situation.

<u>Theorem</u> 4.3. $H_{x_0} : \mathcal{U} \to \mathcal{Y}$ has a continuous left inverse if and only if $d_0 = \text{rank}\, D(t) = m$ for $\forall\, t$. In this case H_{x_0} has a unique left inverse denoted by ${}_LH_{x_0}^{-1}$. So the unique left inverse to the system (4.2-8) and (4.2-9) is given by

$$
{}_LH_{x_0}^{-1} : \begin{cases} \dot{x}(t) = [A(t)-B(t)D^T(t)C(t)]x(t) + B(t)D^T(t)y(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad x(t_0) = x_0 & (4.2\text{-}10) \\ \\ u(t) = D^T(t)y(t) - D^T(t)C(t)x(t) & (4.2\text{-}11) \end{cases}
$$

where $D^{\dagger} = (D^T(t)D(t))^{-1} D^T(t)$ and $m < \ell$.

<u>Theorem</u> 4.4. $H_{x_0} : \mathcal{U} \to \mathcal{Y}$ has at least one continuous right inverse if and only if $d_0 = \text{rank}\, D(t) = \ell$ for $\forall\, t$, and it is denoted by ${}_RH_{x_0}^{-1}$.

If $d_0 = \text{rank}\, D(t) \neq \ell$ for some $t \in [t_0, t_f]$. Then H_{x_0} does not have a right inverse (continuous or not).

<u>Theorem</u> 4.5. $H_{x_0} : \mathcal{U} \to \mathcal{Y}$ has a unique continuous inverse if and only if $d_0 = \text{rank}\, D(t) = m = \ell$ for $\forall\, t$. The inverse of the system (4.2-8) and (4.2-9) in the case that $D(t)$ is a square matrix with full constant rank is expressed as:

$$
H_{x_0}: \begin{cases} \dot{x}(t) = [A(t)-B(t)D^{-1}(t)C(t)]x(t) + B(t)D^{-1}(t)y(t) \\ \qquad\qquad x(t_0) = x_0 & (4.2\text{-}13) \\ u(t) = D^{-1}(t)y(t) - D^{-1}(t)C(t)x(t) & (4.2\text{-}14) \end{cases}
$$

In cases above, $D(t)$ had full and constant rank m or ℓ. Now we consider cases that $m \times m$ matrix $D(t)$ has constant rank but $d_0 < m$ or the case where $\ell \times m$ matrix $D(t)$ has $d_0 = \text{rank}\, D(t) < \min(\ell, m)$.

4.2.3 Inversion Algorithm for Construction of the Inverse System

The basis of the Silverman's algorithm(51) exploits the following theorems by Dolezal.

If $D(t)$ is a matrix with a constant rank $d_0 < \min(m, \ell)$ on $[t_0, t_f]$ and differentiable then there exists a square non-singular matrix $S_0(t)$ such that

$$
S_0(t)D(t) = \begin{bmatrix} \bar{D}_0(t) \\ \hline 0 \end{bmatrix} \qquad (4.2\text{-}15)
$$

where $S_0(t)$ is $m \times m$, $m < \ell$ and $\bar{D}_0(t)$ has d_0 rows and rank $\bar{D}_0(t) = d_0$ on $[t_0, t_f]$.

Defining a system S_0 as

$$
S_0: \begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) & (4.2\text{-}16) \\ y_0(t) = C_0(t)x(t) + D_0(t)u(t) & (4.2\text{-}17) \end{cases}
$$

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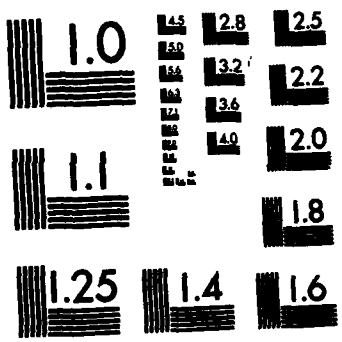

MICROCOPY RESOLUTION TEST CHART
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where

$$y_0(t) = S_0(t)y(t) = \begin{bmatrix} \bar{y}_0(t) \\ \tilde{y}_0(t) \end{bmatrix}, \quad C_0(t) = S_0(t)C(t) = \begin{bmatrix} \bar{C}_0(t) \\ \tilde{C}_0(t) \end{bmatrix}$$

(4.2-18)

The bar and the tilde represent respectively the first d_0 rows and the last $m-d_0$ rows of the matrices $C_0(t)$ and $y_0(t)$.

Now define a matrix differential operator M_0

$$M_0 \equiv \left[\begin{array}{c|c} Id_0 & 0 \\ \hline 0 & I_{m-d_0}\frac{d}{dt} \end{array}\right] \tag{4.2-19}$$

where Id_0 is an identity matrix with order d_0. Then

$$M_0 y_0(t) = \begin{bmatrix} \bar{y}_0(t) \\ \frac{d}{dt}\tilde{y}_0(t) \end{bmatrix} = \begin{bmatrix} \bar{C}_0(t) \\ \tilde{C}_0(t)A(t)+\dot{\tilde{C}}_0(t) \end{bmatrix} x(t) + \begin{bmatrix} \bar{D}_0(t) \\ \tilde{C}_0(t)B(t) \end{bmatrix}$$

(4.2-20)

Consider

$$d_1 = \text{rank} \begin{bmatrix} \bar{D}_0(t) \\ \tilde{S}_0(t)B(t) \end{bmatrix} \tag{4.2-21}$$

if $d_1 = m$, then the inverse is found as before. If $d_1 < m$ then we find $S_1(t)$ such that

$$D_1(t) = C_1(t)\begin{bmatrix} \bar{D}_0(t) \\ \tilde{C}_0(t)B(t) \end{bmatrix} = \begin{bmatrix} \bar{D}_1(t) \\ \hdashline 0 \end{bmatrix} \tag{4.2-22}$$

and the procedure is continued, then by induction we get

$$\mathcal{S}_k: \begin{cases} \dot{x}(t) + A(t)x(t) + B(t)u(t) & (4.2\text{-}23) \\ y_k(t) = C_k(t)x(t) + D_k(t)u(t) & (4.2\text{-}24) \end{cases}$$

where

$$y_k(t) = \begin{bmatrix} \bar{y}_k(t) \\ \tilde{y}_k(t) \end{bmatrix}, \quad D_k(t) = \begin{bmatrix} \bar{D}_k(t) \\ \hdashline 0 \end{bmatrix} \tag{4.2-25}$$

in which $\bar{D}_k(t)$ has d_k rows with rank $\bar{D}_k = d_k$

$$C_k(t) = \begin{bmatrix} \bar{C}_k(t) \\ \tilde{C}_k(t) \end{bmatrix}$$

and $\bar{C}_k(t)$ has d_k rows and $\tilde{C}_k(t) = \tilde{C}_{k-1}(t), A(t) + C_{k-1}(t)$ has $m-d_k$ rows. If $d_k < m$ then

$$M_k = \left[\begin{array}{c|c} I d_k & 0 \\ \hdashline 0 & I_{m-d_k}\frac{d}{dt} \end{array}\right] \tag{4.2-26}$$

and

$$M_k y_k = \begin{bmatrix} \bar{y}_k(t) \\ \frac{d}{dt}\tilde{y}_k(t) \end{bmatrix} = \begin{bmatrix} \bar{C}_k(t) \\ \tilde{C}_{k+1}(t) \end{bmatrix} x(t) + \begin{bmatrix} \bar{D}_k(t) \\ \tilde{C}_k(t)B(t) \end{bmatrix} u(t) \tag{4.2-27}$$

where

$$\tilde{C}_{k+1}(t) = \tilde{C}_k(t)A(t) - \dot{\tilde{C}}_k(t) \tag{4.2-28}$$

Now if d_{k+1} if $d_{k+1} = \text{rank}\begin{bmatrix} \bar{D}_k(t) \\ \tilde{C}_k(t)B(t) \end{bmatrix} < m$ and constant for $t \in [t_0, t_f]$. Then an $m \times m$ non-singular differentiable $S_{k+1}(t)$ can be found such that:

$$D_{k+1}(t) = S_{k+1}(t)\begin{bmatrix} \bar{D}_k(t) \\ \tilde{C}_k(t)B(t) \end{bmatrix} = \begin{bmatrix} \bar{D}_k(t) \\ \hline 0 \end{bmatrix} \tag{4.2-29}$$

where $D_{k+1}(t)$ has d_{k+1} rows and rank $D_{k+1}(t) = d_{k+1}$ and then the system S_{k+1} is defined as

$$S_{k+1}: \begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) & (4.2\text{-}30) \\ y_{k+1}(t) = C_{k+1}(t)x(t) + D_{k+1}(t)u(t) & (4.2\text{-}31) \end{cases}$$

where

$$y_{k+1}(t) = S_{k+1}(t)M_k y_k(t) \tag{4.2-32}$$

and

$$C_{k+1}(t) = S_{k+1}(t) \begin{bmatrix} \bar{C}_k(t) \\ \tilde{C}_k(t)A(t) \end{bmatrix} \begin{bmatrix} \bar{C}_{k+1}(t) \\ \tilde{C}_{k+1}(t) \end{bmatrix} \qquad (4.2\text{-}33)$$

Also $\bar{C}_{k+1}(t)$ has d_{k+1} rows and $\tilde{C}_{k+1}$ has $m-d_{k+1}$ rows.

Suppose now that there exists an integer α such that D_α has rank m. Then it is possible to get the inverse in the form

$$g^{-1}: \begin{cases} \dot{x}(t) = [A(t) - B(t)D_\alpha^T(t)C(t)]x(t) + B(t)D_\alpha(t)y_\alpha(t) \qquad x(t_0)=x_0 & (4.2\text{-}34) \\ u(t) = D_\alpha^T(t)y(t) - D_\alpha(t)C(t)x(t) & (4.2\text{-}35) \end{cases}$$

where

$$D_\alpha^\dagger(t) = (D_\alpha^T(t)D_\alpha(t))^{-1}D_\alpha^T(t) \text{ and } y_\alpha(t) = (\mathop{M}_{i=0}^{\alpha} s_{\alpha-1}\cdot M_{\alpha-i-1})y(t).$$

If $\ell < m$, then, $D_\alpha^\dagger(t) = D_\alpha^T(t)(D_\alpha(t)D_\alpha^T(t))^{-1}$ and if $\ell = m$ the inverse system is found as the form of (4.2-13) and (4.2-14).

Now, a more precise definition of the regular system is the case when matrices $D(t)$ and $[\bar{D}_k(t) \; (\tilde{C}_k(t)B(t)^T]^T$ have constant rank on $t \in [t_0,t_f]$ for $k = 0,1,\ldots,n-1$.

So it is concluded that the regular system representation (4.2-8) and (4.2-9) is invertible if there exists a positive integer $\alpha < n$ such that $d_\alpha = m = \ell$. In the case $d_\alpha = m < \ell$ the system is called left invertible and in the case $d_\alpha = \ell < m$ it is called right invertible. References (54) and (48) show the necessity proof of the above results.

4.3 Determination of Opponent Strategy

In this section we use the results of previous sections to verify the invertibility of the system (2.6-2) and if it exists to construct it and determine control v. We assume each player has perfect information about the output or the state of the system.

Let u^*, v^* and x^* be the saddle point solution of the differential game (2.7-1) to (2.7-4) then we have

$$\dot{x}^*(t) = A(t)x^*(t) + B(t)u^*(t) + C(t)v^*(t) \tag{4.3-1}$$

$$y^*(t) = x^*(t) \tag{4.3-2}$$

$$x(t_0) = x_0 \tag{4.3-3}$$

If $\Delta x(t)$ is the deviation of the saddle point state trajectory due to the deviation of the evader's control $\Delta v(t)$ and $\Delta u(t)$ is the change of pursuer's control in response to $\Delta v(t)$, we will have

$$\left.\begin{aligned} v(t) &= v^*(t) + \Delta v(t) \\ u(t) &= u^*(t) + \Delta u(t) \\ x(t) &= x^*(t) + \Delta x(t) \\ y(t) &= y^*(t) + \Delta y(t) \end{aligned}\right\} \quad \forall\, t \in [t_0, t_f] \tag{4.3-4}$$

From (4.3-1) and (4.3-4) we get

$$\Delta\dot{x}(t) = A(t)\Delta x(t) + B(t)\Delta u(t) + C(t)\Delta v(t) \qquad (4.3\text{-}5)$$

$$\Delta y(t) = \Delta x(t) \qquad (4.3\text{-}6)$$

$$\Delta x(t_0) = 0 \qquad (4.3\text{-}7)$$

We assume the output measurement is taken by the pursuer periodically in a small time interval $\Delta t = t^{i+1} - t^i$, $i = 0,1,\ldots$. When the game starts running, the pursuer takes a measurement at $t_1 = t_0 + \Delta t$ and compares this value with the saddle point solution and notices the difference $\Delta y(t_1) = \Delta x(t_1)$ which is caused by non-optimal playing of the evader's strategy. Since at this time the pursuer knows his own strategy

$$\Delta u(t) = 0 \quad \text{at} \quad t = t_1 \qquad (4.3\text{-}6)$$

so at any time of the measurement

$$\Delta\dot{x}(t) = A(t)\Delta x(t) + C(t)\Delta v(t) \qquad (4.3\text{-}7)$$

$$\Delta x(t_0) = 0 \qquad (4.3\text{-}8)$$

$$\Delta y(t) = \Delta x(t) \qquad (4.3\text{-}9)$$

Assuming invertibility criteria of the Section 4.2, i.e., Theorem 4.3 for the linear system (4.3-7) - (4.3-9) hold. Since there is perfect information, by measuring the output and by knowing the saddle point at each time $\Delta x(t_1)$ is computable by the pursuer. Having a sufficient history of observation time derivatives of $\Delta x(t)$ can be computed at each time by forward difference approximation. Hence $\Delta v(t_1)$ is determined at each instant of measurement using (4.2-34) and (4.2-35). So, at this time it is assumed that the evader continues the game with the same deviation from the saddle point trajectory until the end of the game.

$$v(t) = v^*(t) + \Delta v(t_1) \quad \text{for} \quad \forall\, t \in [t, t_f] \tag{4.3-10}$$

Now in this time interval $v(t)$ is considered as an external input to the system and a one-sided singular optimal control problem with initial condition

$$x(t_1) = x^*(t_1) + \Delta x(t_1) \tag{4.3-11}$$

is solved by applying the proposed technique for singular problems. The control u is obtained periodically at each time. Again at time $t_2 = t_1 + \Delta t$ another measurement is taken and compared with the recent updated trajectory and procedure is continued until the game is terminated. By this technique a suboptimal solution can be obtained.

If the evader's deviation from the saddle point solution is sufficiently small then the pursuer's sequence of singular and non-singular control will be the same. In this case the location of switching time and the value of singular control will be changed. If the deviation is large enough the sequence of singular and non-singular controls will be changed.

In non-linear problems the system equations are linearized around a known reference trajectory (usually saddle point trajectory).

$$\dot{x} = f(x,u,v) \tag{4.3-14}$$

$$x(t_0) = x_0 \tag{4.3-15}$$

$$y(t) = x(t) \tag{4.3-16}$$

Assuming $f(x,u,v)$ is continuous in x, u and v the linearized equations are

$$\delta\dot{x} = f_x\delta x + f_u\delta u + f_v\delta v \tag{4.3-17}$$

$$\delta x(t_0) = 0 \tag{4.3-18}$$

where

$$f_x = \left.\frac{\partial f}{\partial x}\right|_* , \quad f_u = \left.\frac{\partial f}{\partial u}\right|_* \text{ and } f_v = \left.\frac{\partial f}{\partial v}\right|_* \tag{4.3-19}$$

Taking measurement of the output at some time interval $\Delta t = t^{i-1} - t^i$, $i = 0,1,\ldots$ and approximating

$$\left.\begin{aligned} \delta x &= x_{real} - x_{ref} \\ \delta u &= u_{real} - u_{ref} \\ \delta v &= v_{real} - v_{ref} \\ \delta y &= y_{real} - y_{ref} \end{aligned}\right\} \quad \forall\, t \in [t_0,t_f] \tag{4.3-20}$$

similar to the linear case at each time of the measurement

$$\delta u = 0 \tag{4.3-21}$$

therefore

$$\begin{cases} \delta\dot{x} = f_x \delta x + f_v \delta v & (4.3\text{-}22) \\ \delta x(t_0) = 0 & (4.3\text{-}23) \\ \delta y = \delta x & (4.3\text{-}24) \end{cases}$$

Assuming the inverse to (4.3-22) - (4.3-24) exists the procedure will be similar to linear case.

Note: This method requires relinearization and it is assumed that the deviation is slight and sharp deviation does not occur frequently. Although this does not restrict the method but practically reduces the accuracy of the technique.

Anderson(40,41) has proposed a near optimal method by taking measurement of the states at equally time interval Δt and updating the set of TPBVP obtained from the necessary conditions for the saddle point. But since the solutions obtained by this procedure are based on the assumption that both players play optimally the solution cannot be as good as the case that one player knows the opponent strategy.

Jachinovitz(47) used estimation techniques to determine the evader's control through state measurement and achieved a better performance than Anderson. But, his input estimation algorithm was very time consuming so that it was impractical for on line purposes. By inverse system technique input determination is done very fast so it is convenient for on line use.

4.4 On Line Solution of the Example

We consider the numerical example in Section 3.5. Suppose the evader due to the biased error in his control has some deviation from the saddle point trajectory. Assume this deviation is $\Delta v = .1$ such that $v(t) = v^*(t) + .1$ for $0 \le t \le 3.17$ where $v^*(t)$ is his saddle point solution. This control $v(t)$ is not known to the pursuer unless he determines it by direct observation and measurement of the output or state of the system. In this example we assume there are perfect measurements and $y(t) = x(t)$.

At time $t = 0$ the system starts running in real time. Some measurements are taken at some small time intervals and compared with the saddle point trajectories. Consider the system (4.3-7) - (4.3-9), since rank $[C] = 1$ for this example, from Theorem 4.3 the system is exists. So, the evader's input deviation $\Delta v(t)$ is found from the following relationship:

$$\Delta v(t) = (C^T C)^{-1} C^T \Delta \dot{x}(t) - (C^T C)^{-1} A \Delta x \qquad (4.4\text{-}1)$$

$$\Delta y(t) = \Delta x(t) \qquad (4.4\text{-}2)$$

In the following we find the difference between the saddle point trajectory (starred quantities) and real trajectory.

$t = 0$	$x_1^* = .59847$	$x_2^* = .60706$
$t = 0$	$x_1 = .59847$	$x_2 = .60706$

and $\Delta v(.1) = 0.102$. Taking another measurement at time $t = 0.2$ we will get

$t = .2$	$x_1^* = .70009$	$x_2^* = .40916$
$t = .2$	$x_1 = .70420$	$x_2 = .42956$

So,

$t = .2$	$\Delta x_1 = .00411$	$\Delta x_2 = .02040$
$t = .2$	$\Delta \dot{x}_1 = .122$	$\Delta \dot{x}_2 = .10200$

and $\Delta v(.2) = .1020$. It is noticed that the computed deviation is .102 from the optimal open loop strategy. Therefore, using the external control $v(t) = v^*(t) + .102$ over $.2 \leq t \leq 3.17$ with the initial condition $x_1(.2) = .70420$ and $x_2(.2) = .42956$ a one-sided optimal control problem results. This problem was solved and the pursuer's control was computed as

$$u = -1 \quad \text{for} \quad 0 \leq t \leq 1.33$$

$$u = u_{singular} \quad \text{for} \quad 1.33 < t \leq 3.17$$

$$\text{where} \quad u_{singular} = x_1(t) - \Delta v(t) = x_1(t) - .1$$

Tables (4.1-a) and (4.1-b) show the computer results of the computations.

As long as measurements show that $\Delta v \simeq .1$ we may consider this to be the optimal solution.

Now in order to show the advantage of the method we compare the results of the following 3 cases.

Case 1. Two players play optimal open loop (saddle point strategies).

Case 2. The evader deviates from his saddle point strategy but the pursuer plays his optimal open loop strategy.

Case 3. The evader deviates from his saddle point strategy and the pursuer plays his optimal closed loop strategy.

The following tables show the results of the computation for these cases.

$\Delta v(t)$	t_s	J
0	1.2	.60243

Table (4.1). Case 1. Two players play optimal open loop (saddle point).

The computer print out of Case 1 is given in Table (3.2-a,b).

For the Case 2 the results are

$\Delta u(t)$	t_s	J
+.1	1.2	-.86687
-.1	1.2	-.81053

Table (4.2) Case 2. The evader deviates and pursuer plays his optimal open loop.

The computer printout of this case for $\Delta v(t) = .1$ is given in Table (4.4-a) and (4.4-b) and Figures (4.1-4.2) show the controls and trajectories.

For the Case 3 the results are

$\Delta v(t)$	t_s	J
+.1	1.33	-.99045
-.1	1.09	-1.02390

Table (4.3) Case 3. The evader deviates and the pursuer uses his optimal closed loop strategy dependent control.

The computer printout of this case is given in Table (4.5-a,b) and Figures (4.3-4.4) show the controls and trajectories.

As we notice from the example, for the small deviation of the control $\Delta v(t) = .1$ the sequence of the pursuer's control is the same but the swtiching time is increased from 1.2 to 1.33. Also the value of singular control is slightly changed (Figure 4.3-a,b).

As the results in Tables (4.1) and (4.2) show in Case 2, the pursuer will achieve a better performance than Case 1. The results in Table (4.3) show that in Case 3 the pursuer can achieve even a better performance than Case 2.

We can also consider the cases that the evader's deviation does not remain constant, but changes at some times. Two cases are considered. In the first case the evader deviates $\Delta v(t) = .1$ for $0 \leq t \leq .605$ and then from time $t = .605$ on plays saddle point strategy until the game terminates.

In the second case $\Delta v = .1$ for $0 \leq t \leq .605$ and $\Delta v = .05$ for $.605 < t \leq 3.17$.

In the following, again we calculate the control deviation Δv by using the inverse system technique. For

For $0 \leq t \leq .605$ we already computed Δv and in this interval Δv was computed at the instant of time which was constant $\Delta u(t) = .102$.

At time $t = .605$ measurement was taken and comparison was made with the new saddle point reference trajectory. The results are

$t = .605$	$x_1^* = .80297$	$x_2^* = .06794$
$t = .605$	$x_1 = .80297$	$x_2 = .06794$

and in the next measurement we get

Time	x_1	x_2
0.0	0.59847	0.60706
0.05000	0.62771	0.56268
0.10000	0.65474	0.51830
0.15000	0.67954	0.47389
0.20000	0.70212	0.42938
0.25000	0.72248	0.38489
0.30000	0.74061	0.34029
0.40000	0.77022	0.25107
0.50000	0.79086	0.16170
0.60000	0.80255	0.07218
0.70000	0.80579	-0.01746
0.80000	0.79905	-0.10720
0.90000	0.78384	-0.19703
1.00000	0.75965	-0.28693
1.10000	0.72646	-0.37687
1.20000	0.68427	-0.46685

Table (4.4-a) Numerical results for Case 2. (Pursuer uses u_n)

Time	x_1	x_2
1.20000	0.68427	-0.46685
1.25000	0.66181	-0.43134
1.30000	0.64114	-0.39571
1.35000	0.62222	-0.36068
1.40000	0.60501	-0.32690
1.45000	0.58957	-0.29391
1.50000	0.57550	-0.26471
1.55000	0.56316	-0.23465
1.60000	0.55197	-0.20745
1.65000	0.54242	-0.17974
1.70000	0.53395	-0.15445
1.75000	0.52699	-0.12876
1.80000	0.52104	-0.10521
1.85000	0.51648	-0.08131
1.90000	0.51287	-0.05933
1.95000	0.51055	-0.03703
2.00000	0.50913	-0.01645
2.05000	0.50892	0.00442
2.10000	0.50955	0.02377
2.15000	0.51130	0.04337
2.20000	0.51386	0.06162
2.25000	0.51747	0.08009
2.30000	0.52185	0.09738
2.35000	0.52721	0.11487
2.40000	0.53331	0.13131
2.45000	0.54034	0.14793
2.49999	0.54809	0.16365
2.54999	0.55671	0.17953
2.59999	0.56603	0.19463
2.64999	0.57618	0.20987
2.69999	0.58700	0.22444
2.74999	0.59862	0.23915
2.79999	0.61091	0.25331
2.84999	0.62396	0.26757
2.89999	0.63766	0.28140
2.94999	0.65210	0.29532
2.99999	0.66718	0.30890
3.04999	0.68299	0.32256
3.09999	0.69944	0.33599
3.14998	0.71655	0.34948
3.19998	0.73438	0.36281

Table (4.4-b) Numerical results for Case 2 (Pursuer uses u_s)

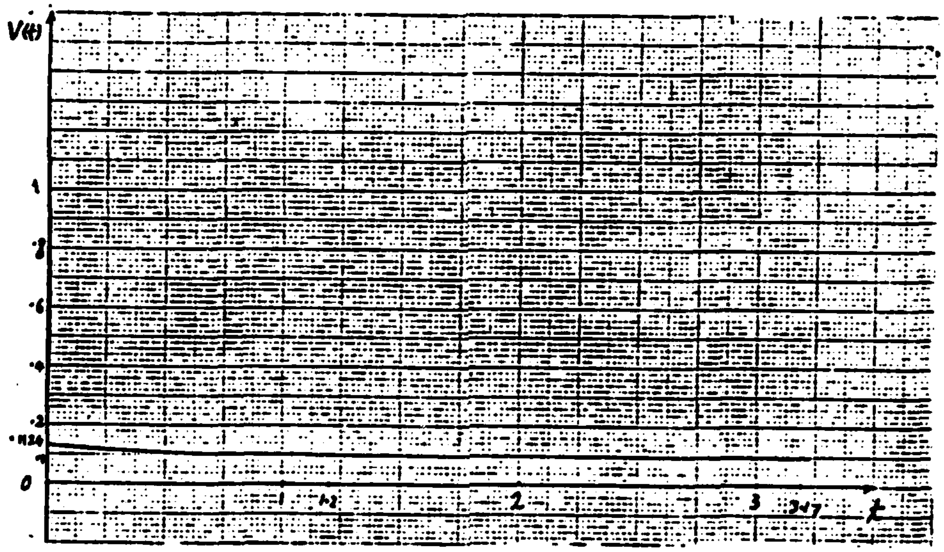

Figure (4.1-a) Deviated evader's control

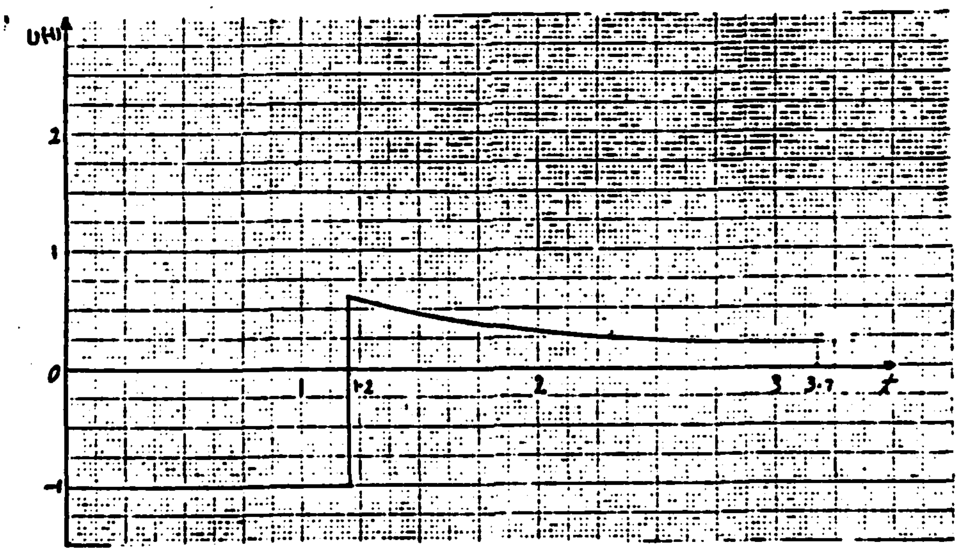

Figure (4.1-b) Pursuer's saddle point control

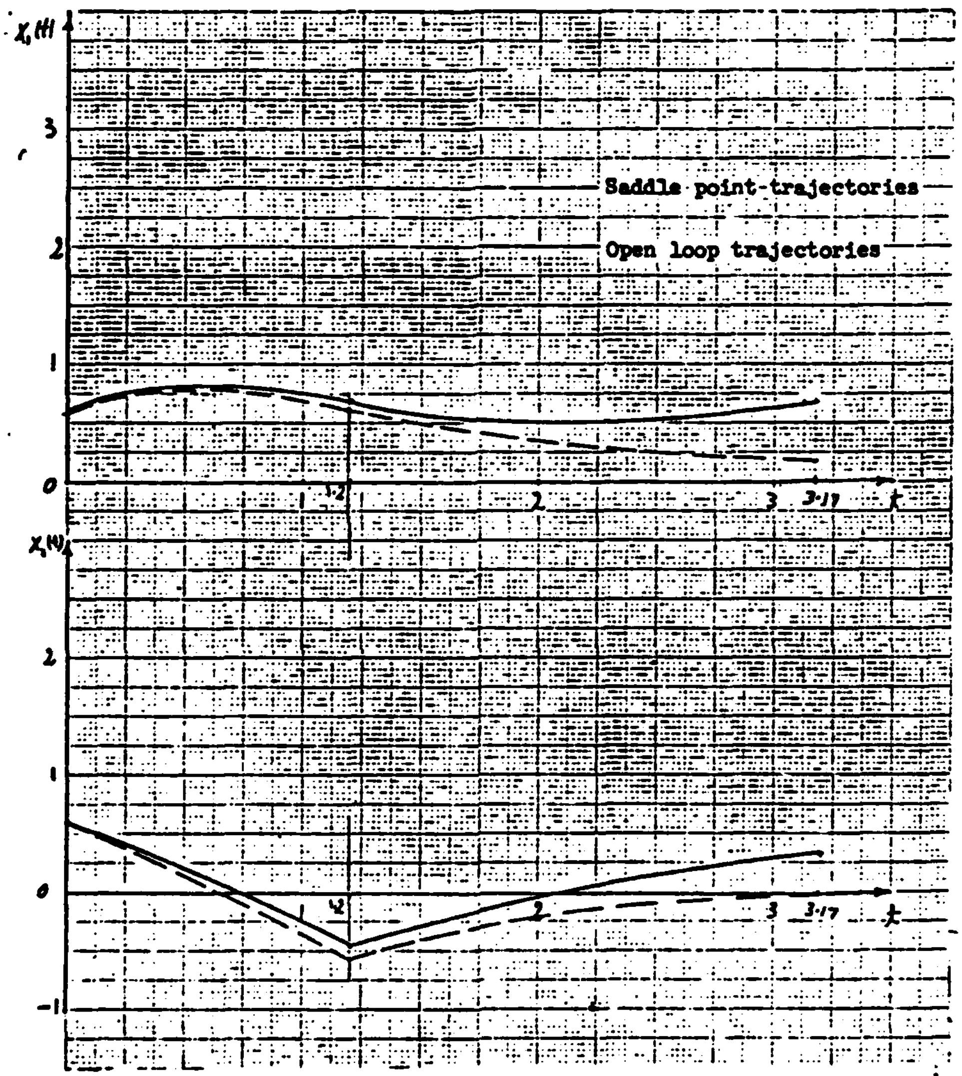

Figure (4.2) Deviated open loop state trajectories

Closed Loop Strategy Dependent Solutions

Time	x_1	x_2	λ_1	λ_2
-0.07000	0.55381	0.66899	1.60729	1.60631
0.03000	0.61629	0.58748	1.54871	1.38598
0.13000	0.66990	0.49177	1.48433	1.18067
0.23000	0.71463	0.40295	1.41503	0.99093
0.33000	0.75047	0.31375	1.34177	0.81723
0.43000	0.77738	0.22409	1.26524	0.65995
0.53000	0.79536	0.13508	1.18653	0.51937
0.63000	0.80439	0.04554	1.10646	0.39568
0.73000	0.80446	-0.04412	1.02595	0.28899
0.83000	0.79557	-0.13388	0.94597	0.19930
0.93000	0.77759	-0.22372	0.86713	0.12655
1.03000	0.75082	-0.31362	0.79063	0.07055
1.08000	0.73402	-0.35859	0.75350	0.04875
1.13000	0.71496	-0.40357	0.71727	0.03104
1.18000	0.69366	-0.44856	0.68205	0.01737
1.23000	0.67011	-0.49355	0.64794	0.00767
1.28000	0.64430	-0.53855	0.61507	0.00191
1.33000	0.61625	-0.58355	0.58355	0.0

Table (4.5-a) Numerical results - Non-singular arc

Time	x_1	x_2	λ_1	λ_2
1.33000	0.61625	-0.58355	0.58355	0.0
1.38000	0.58783	-0.55345	0.55345	0.0
1.43000	0.56088	-0.52474	0.52474	0.0
1.48000	0.53533	-0.49734	0.49734	0.0
1.53000	0.51113	-0.47119	0.47119	0.0
1.58000	0.48820	-0.44621	0.44621	0.0
1.63000	0.46649	-0.42235	0.42235	0.0
1.73000	0.42652	-0.37773	0.37773	0.0
1.83000	0.39081	-0.33690	0.33690	0.0
1.93000	0.35902	-0.29944	0.29944	0.0
2.03000	0.33053	-0.26498	0.26498	0.0
2.13000	0.30594	-0.23316	0.23316	0.0
2.23000	0.28411	-0.20369	0.20369	0.0
2.33000	0.26513	-0.17625	0.17625	0.0
2.43000	0.24881	-0.15057	0.15057	0.0
2.53000	0.23497	-0.12640	0.12640	0.0
2.63000	0.22348	-0.10350	0.10350	0.0
2.73000	0.21473	-0.08163	0.08163	0.0
2.83000	0.20713	-0.06058	0.06058	0.0
2.93000	0.20210	-0.04014	0.04014	0.0
3.03000	0.19959	-0.02029	0.02029	0.0
3.13000	0.19807	-0.00025	0.00025	0.0
3.27999	0.19964	0.01959	-0.01959	0.0

Table (4.5-b) Numerical results - singular arc

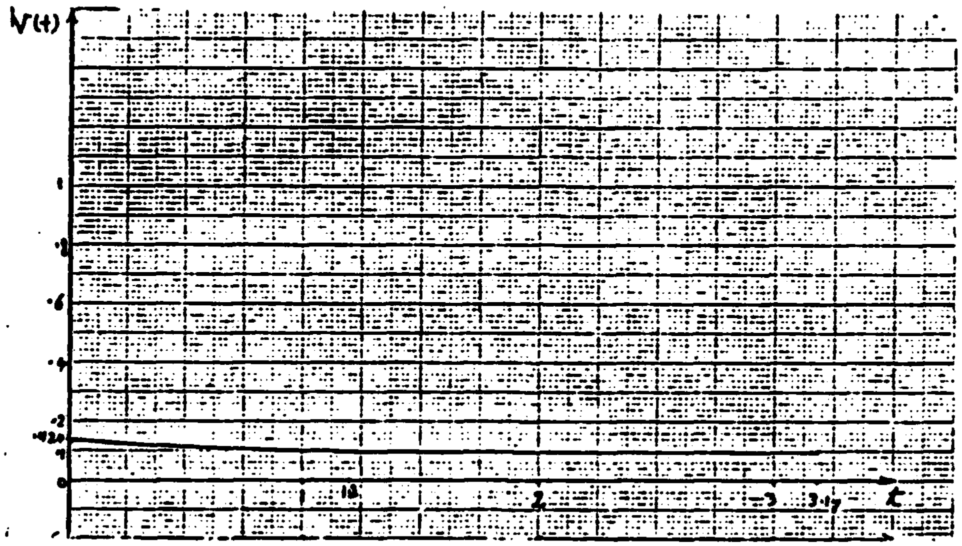

Figure (4.3-a) Deviated evader's control

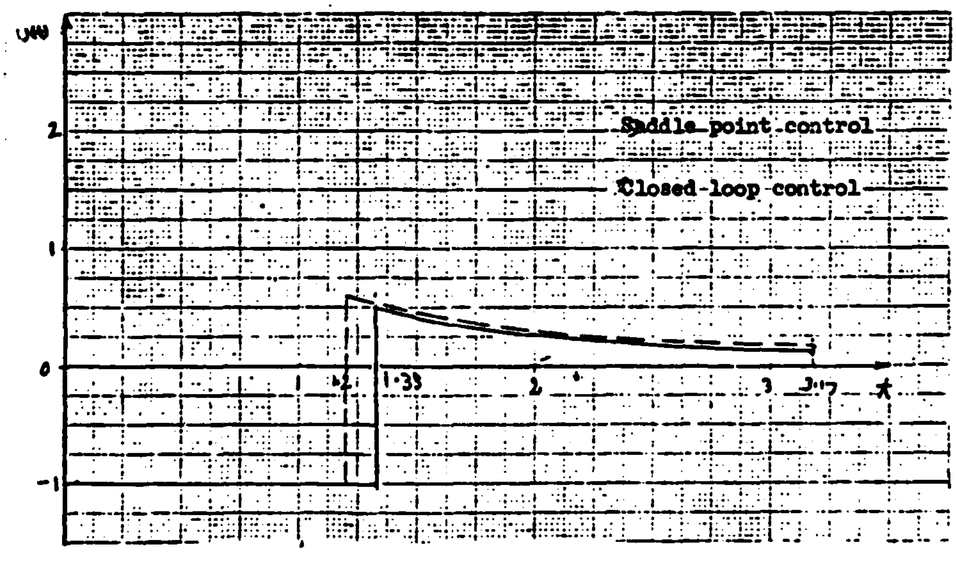

Figure (4.3-b) Closed loop strategy dependent Pursuer's control

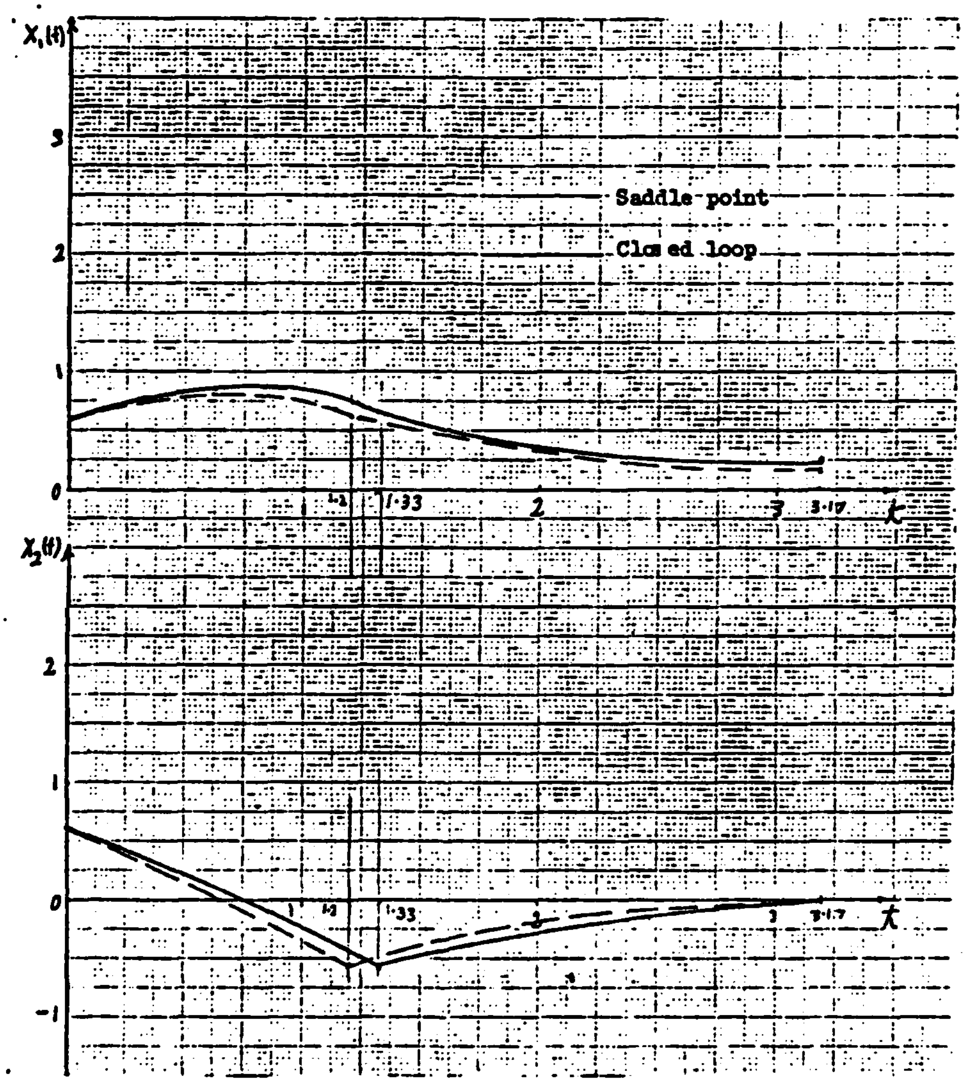

Figure (4.4) Closed loop strategy dependent state trajectories

$t = .700$	$x_1^* = .80493$	$x_2^* = -.02674$
$t = .700$	$x_1 = .80515$	$x_2 = -.02197$

So,

$t = .700$	$\Delta x_1 = .00022$	$\Delta x_2 = .00477$
$t = .700$	$\Delta \dot{x}_1 = .00231$	$\Delta \dot{x}_2 = .05210$

and from (4.4-1) $\Delta v(.7) = .052$.

By taking another measurement at time $t = .75$ we have

$t = .750$	$x_1^* = .80236$	$x_2^* = -.07661$
$t = .750$	$x_1 = .80287$	$x_2 = -.06934$

and the deviation in evader's control is $\Delta v(.75) = .05$.

By this procedure $\Delta v = .05$ is obtained and if we continue taking measurement and calculating Δv, we will see $\Delta v = .05$ for the rest of the interval.

Tables (4.6) and (4.7) show the performance and switching times for Case 2 and Case 3 when the evader changes his deviation during the game.

$\Delta v(t)$	t_s	J
.1 for $0 \leq t \leq .605$ 0 for $.605 < t \leq 3.17$	1.2	.27519
.1 for $0 \leq t \leq .605$ .05 for $.605 \leq t \leq 3.17$	1.2	-.05795

Table (4.6) The evader deviates and the pursuer plays optimal open loop

Δv	t_s	J
.1 $0 \geq t \leq .605$ 0 $.605 \leq t \leq 3.17$	1.277	.21389
.1 $0 \leq t \leq .605$ .05 $.605 \leq t \leq 3.17$	1.3	-.09583

Table (4.7) The evader deviates and the pursuer plays optimal closed loop strategy dependent

CHAPTER 5

DISCUSSION, CONCLUSION AND RECOMMENDATION

5.1 Discussion and Conclusion

This dissertation is mainly concerned with singularity in differential game problems with linear systems. A class of generalized pursuit-evasion games with linear state equations and bounds on the control in which Hamiltonian is linear in pursuer's control was introduced. This problem for some values of initial conditions and final times can have an optimal solution with singular interval. A sufficient condition for the existence of a saddle point has been obtained. One of the advantages of this condition is that an extremal value for each one of the five parameters of the game, i.e., t_f, S, R, Q and C can be determined through the inequality (2.6-18) (provided that all other four parameters are known) to guarantee the existence of a unique saddle point.

Due to the constraint on the control, in general, there is no analytical solution for this class of games. Rapid and efficient numerical techniques are required to solve physical and practical problems with singular arcs and controls with discontinuities. An indirect numerical method was proposed here, that could generate an accurate and fast solution to a class of singular problems with linear systems in which Terminal Error Function is only function of switching times. In this method the sequence of singular and non-singular arcs is estimated by physical or mathematical insight to the problem.

Newton's method was used to iterate the T.E.F. on the switching time. By linearization this technique was extended to solve the same class of problems with non-linear systems. For a broader class of problems with linear systems in which T.E.F. is a function of both switching times and initial costates a second approach was proposed. In this approach we iterate T.E.F. on both initial costates and switching times. As was considered in Chapter 3, the singular problems with order of higher than one, and also problems with bang bang controls could be solved through the second approach. In singular problems with non-linear systems, since the junction conditions are linearized to the first order, they may not be satisfied precisely at the same time.

Because of accuracy and rapid convergence, the proposed technique is superior to some other numerical techniques for singular optimal problems, e.g., Gradient Method, Epsilon Method or Quasilinearization-Epsilon Method. Sometimes the approximate solutions generated by some of these methods may provide information to estimate the sequence of controls and switching times.

A numerical example was solved using Newton's technique and with an appropriate initial guess, for the switching time, the solution converged after six iterations.

By applying the closed-loop strategy dependent method some numerical solutions were obtained for the same example. From the numerical results we considered that if the evader deviates from his optimal open loop strategy, the pursuer can perform better, provided he uses closed-loop strategy dependent policy. For applying this kind of strategy the existence and stability of the inverse system is required. The opponent's strategy at each instant of time can be rapidly determined by inverse systems so that it is very appropriate for on-line use. In this kind of closed loop solution, state or output measurement should be taken periodically in order to generate an approximate solution. If the measurement interval Δt gets smaller, a better and more accurate solution is obtained. However, as Δt gets smaller the computation time increases. Some considerations should be taken on the choice of Δt so that computational barrier is not encountered. This interval should be smaller than the smallest time constant of the system and also should be larger than computation time at each period.

5.2 Recommendations

The study of the literature in singular optimal control problems shows that still much research needs to be done on the problems of computation of optimal singular control. The analysis of the junction points should be studied further and some efforts are required for locating the singular arcs and obtaining a sufficient condition for partially singular problems.

A general and accurate method for solving non-linear problems with all orders of singularities is not yet available.

The present work stimulates further development for computational research in optimal singular control and differential game problems.

The class of differential games in which singularity may occur for both pursuer's and evader's control is an interesting topic for computational research.

In the case above closed loop strategy dependent solution is another aspect of the research which possibly encounters the difficulty of discontinuities in controls.

The extension of the differential dynamic programming-Epsilon method or Quasilinearization-Epsilon method to differential game problems seems to be advantageous to overcome the difficulty of estimating the sequence of controls and initial switching times.

The extension of the proposed techniques to free final time problems with terminal manifolds is suggested for further research.

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APPENDIX A

A.1 Proof of Theorem 2.1

Here we shall derive GLC necessary condition $(-1)^q[d^{2q}/dt^{2q}\, H_u] \geq 0$ for the case when $q = 1$.

Considering (2.1-1) to (2.1-3) from neighboring extremal theory we will have

$$\delta\dot{x} = f_x \delta x + f_u \delta u \tag{A.1-1}$$

$$\delta\dot{\lambda} = -H_{xx}\delta x + f_u^T\, \delta\lambda + H_{xu}\lambda u \tag{A.1-2}$$

$$\delta x(t_0) = 0 \tag{A.1-3}$$

$$\delta\lambda(t_f) = h_{xx}\delta x|_{t=t_f} \tag{a.1-4}$$

From the second variation

$$\delta^2 J = \frac{1}{2}\,[\delta x^T h_{xx}\delta x]_{t_f} + \frac{1}{2}\int_{t_0}^{t_f} [\delta x^T \delta u^T] \begin{bmatrix} H_{xx} & H_{xu} \\ U_{ux} & H_{uu} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta u \end{bmatrix} dt$$

(A.1-5)

subject to:

$$\begin{cases} \delta\dot{x} = H_{\lambda x}\delta x + H_{\lambda u}\delta u \qquad \delta x(t_0) = 0 & \text{(A.1-6)} \\ \delta\dot{\lambda} = -H_{x\lambda}\delta\lambda - H_{xx}\delta x - H_{xu}\delta u & \text{(A.1-7)} \end{cases}$$

From (A.1-7) it is obvious that

$$\frac{1}{2}\int_{t_0}^{t_f}[\delta\dot{\lambda}^T + \delta\lambda^T f_x + \delta u^T H_{ux} + \delta x^T H_{xx})\delta x dt = 0 \qquad \text{(A.1-8)}$$

Integrating the first term by parts we obtain

$$\frac{1}{2}\int_{t_0}^{t_f}[-\delta\lambda^T\delta\dot{x} +)\delta\lambda^T f_x + \delta u^T H_{ux} + \delta x^T H_{xx})\delta x]dt + \frac{1}{2}[\delta\lambda^T\delta x]_{t_0}^{t_f} = 0$$

(A.1-9)

From (A.1-1) and (A.1-4)

$$\frac{1}{2}\int_{t_0}^{t_f}[-\delta\lambda^T H_{\lambda u}\delta u + \delta u^T H_{ux}\delta x + \delta x^T H_{xx}\delta x]dt + \frac{1}{2}[\delta x^T h_{xx}\delta x]_{t_f} = 0$$

(A.1-10)

Subtracting Equation (A.1-10) from Equation (A.1-5) and simplifying gives

$$\delta^2 J = \frac{1}{2}\int_{t_0}^{t_f}(\delta x^T H_{xu} + \delta\lambda^T H_{\lambda u} + \delta u^T H_{uu})\delta u dt \qquad \text{(A.1-11)}$$

By differentiating the inside of the parenthesis and substituting from (A.1-6) and (A.1-7) we will get

$$\frac{d}{dt}(\delta x^T H_{xu} + \delta\lambda^T H_{\lambda u} + \delta u^T H_{uu}) = \delta x^T(\dot{H}_u)_x + \delta\lambda^T(\dot{H}_u)_\lambda + \delta u^T(\dot{H}_u)_u$$

(A.1-12)

$$\frac{d}{dt}(\delta x^T H_{xu} + \delta u^T H_{\lambda u} + \delta u^T H_{uu}) = \delta x^T(\ddot{H}_u)_x + \delta\lambda^T(\ddot{H}_u)_\lambda + \delta u^T(\ddot{H}_u)_u$$

(A.1-13)

Integrating (A.1-11) by parts together with (A.1-12) we get

$$\delta^2 J = -\frac{1}{2}\int_{t_0}^{t_f} [\delta x^T(\dot{H}_u)_x + \delta\lambda^T(\dot{H}_u)_\lambda + \delta u^T(\dot{H}_u)_u]\delta u_1 dt$$

$$+ \frac{1}{2}[(\delta x^T H_{xu} + \delta\lambda^T H_{\lambda u} + \delta u^T H_{uu})\delta u_1]_{t_0}^{t_f}$$

(A.1-14)

where

$$\delta u_1 = \int_{t_0}^{t_f} \delta u(t)dt \qquad \text{(A.1-15)}$$

By similar integration by parts of (A.1-14) and substituting from (A.1-12) and (A.1-13) we will have

$$\delta^2 J = \frac{1}{2}\int_{t_0}^{t_f} [\delta x^T(\ddot{H}_u)_x + \delta\lambda^T(\ddot{H}_u)_\lambda + \delta u^T(\ddot{H}_u)_u]\delta u_2(t)dt$$

$$+ \frac{1}{2}[(\delta x^T H_{xu} + \delta\lambda^T H_{\lambda u} + \delta u^T H_{uu})\delta u_1]_{t_0}^{t_f} - \frac{1}{2}[\delta x^T(\dot{H}_u)_x + \delta\lambda^T(\dot{H}_u)_\lambda$$

$$+ \delta\lambda^T(\dot{H}_u)_u \delta u_2]_{t_0}^{t_f} \qquad \text{(A.1-16)}$$

$$\delta u_2(t) = \int_{t_0}^{t_f} \delta u_1(t)dt \qquad \text{(A.1-17)}$$

We know in a non-singular problem $H_{uu} \geq 0$ which is a convexity condition. For the singular case where $H_u = H_{uu} = 0$ we consider a special variation as the following.

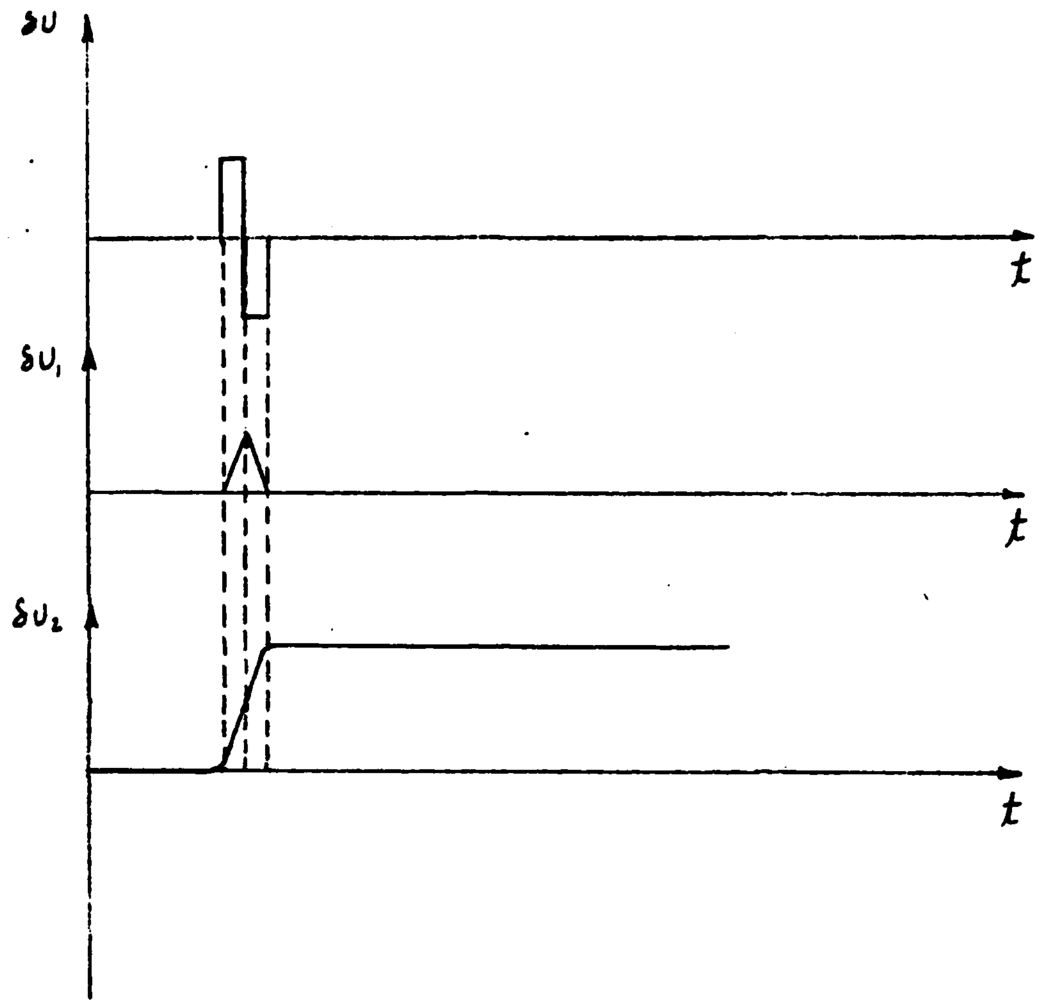

Figure A-1. Special variation in control for deriving GLC conditions

δu (which is chosen to be a positive impulse followed by a negative impluse), δu_1 and δu_2 are shown in Figure (A-1). We know that $\dot{H}_u$ is independent of u, so $(\dot{H}_u)_u = 0$, and as a result of double impulse $\delta x = \delta\lambda = 0$, from which all the terms in (A.1-16) involving δx and $\delta\lambda$ vanish. We take $(\ddot{H}_u)_u$ constant during the period of δu, from the Figure (A-1) it is obvious that

$$\int_{t_0}^{t_f} (\delta u \; \delta u_2) dt < 0 \tag{A.1-18}$$

So, in order to assure $\delta j \geq 0$ it is necessary that

$$(\ddot{H}_u) \leq 0 \tag{A.1-19}$$

This was for the case that u appears explicitly in the second time derivative of H_u. For cases that u appears in the higher order, the procedure of the derivation of the necessary conditions conceptually is the same (see References (20), (31)).

Proof of Theorem 2.2. From (2.1-18) and (2.1-19) by hypothesis we know that $\alpha(t)$ and $\beta(t)$ are continuous and have at least r continuous derivatives at t_s.

Let ε be a small non-zero real number such that $t_s + \varepsilon$ is a point on the non-singular side of t_s and $t_s - \varepsilon$ is a point on the singular side of t_s. Also, the limit of $u^{(i)}(t_s + \varepsilon)$ and $u^{(i)}(t_s - \varepsilon)$ when $\varepsilon \to 0$ are $u_n^{(i)}(t_s)$ and $u_s^{(i)}(t_s)$ respectively.

Define

$$\varphi = H_u \tag{A.2-1}$$

If $K = 2q + r$ then $\varphi^{(k)}$ will be the lowest order derivative of φ which is discontinuous at t_s. We expand $\varphi(t_s + \varepsilon)$ in Taylor series about t_s, and we know $\varphi = 0$ on the singular subarc.

The furst non-zero term of the Taylor series contains the term $\varphi^{(k)}$.

We know

$$\varphi^{(k)} = \frac{d^k}{dt^k}[\alpha + \beta u] \tag{A.2-2}$$

$$\varphi(t_s + \varepsilon) = \frac{\varepsilon^k}{k!}[\alpha^{(r)}(t_s) + \sum_{i=0}^{r} \binom{r}{i}\beta^{(r-i)}(t_s)u_n^{(i)}(t_s)] + 0(\varepsilon^k) \tag{A.2-3}$$

where Leibriz's formula for differentiation of a product has been used to differentiate βu_n.

On the singular subarc we have

$$\varphi^{(2q)} = \alpha + \beta u_s \equiv 0 \tag{A.2-4}$$

So, we will get

$$\alpha^{(r)} = \frac{d^r}{dt^r}[-\beta u_s] = -\sum_{i=0}^{r} \binom{r}{i}\beta^{(r-i)}u_s^{(i)} \tag{A.2-5}$$

By substitution from (A.2-5) into (A.2-3) we will get

$$\varphi(t_s + \varepsilon) = \frac{\varepsilon^k}{k!} \sum_{i=0}^{r} \binom{r}{i}\beta^{(r-i)}(t_s)[u_n^{(i)}(t_s) - u_s^{(i)}(t_s)] + 0(\varepsilon^k) \tag{A.2-6}$$

If $r > 0$

$$u_n^{(i)}(t_s) = u_s^{(i)}(t_s) \qquad i = 0,.,\ldots,r-1 \tag{A.2-7}$$

So, (A.2-6) turns out to be

$$\varphi(t_s + \varepsilon) = \frac{\varepsilon^k}{k!}\beta(t_s)[u_n^{(r)} - u_s^{(r)}] + O(\varepsilon^k) \qquad \text{(A.1-8)}$$

Let

$$\sigma = -\mathrm{sgn}\,\varphi\,(t_s + \varepsilon) \qquad \text{(A.2-9)}$$

$$u_n(t) = \sigma K(t) \qquad \text{(A.2-10)}$$

and

$$u_n^{(i)}(t_s) = \lim_{\varepsilon \to 0} u_n^{(i)}(t_s + \varepsilon) \qquad \text{(A.2-11)}$$

$$u_n^{(i)}(t_s) = \sigma K^{(i)}(t_s) \quad i = 0,.,\ldots,r \qquad \text{(A.2-12)}$$

On the singular arc, the left side can be expended as

$$\sigma K(t_s - \varepsilon) - u(t_s - \varepsilon) = \sum_{i=0}^{r} \frac{(-\varepsilon)^i}{i!}[\sigma K^i(t_s) - u_s^{(i)}(t_s)] + O(\varepsilon)^r \qquad \text{(A.2-13)}$$

By using (A.2-7) and (A.2-12) the right hand side of (A.2-13) can be simplified to

$$\sigma K(t_s - \varepsilon) - u(t_s - \varepsilon) = \frac{(-1)^r\varepsilon^r}{r!}[u_n^{(r)}(t_s) - u_s^{(r)}(t_s)] + O(\varepsilon)^r \qquad \text{(A.2-14)}$$

We substitute from (A.2-13) into (A.2-6) and having $K = 2q + r$ we get

$$\varphi(t_s + \varepsilon) = (-1)^r \frac{\varepsilon^{2q_r}}{K!} \beta(t_s)[\sigma K(t_s - \varepsilon) - u(t_s - \varepsilon)] + O(\varepsilon)^K \tag{A.2-15}$$

From the application of minimum principle on the non-singular subarc $\sigma = 1$ if $\varphi(t_s + \varepsilon) < 0$ and $\sigma = -1$ if $\varphi(t_s + \varepsilon) > 0$. Therefore we have

$$(-1)^r \varepsilon^{2q} \beta(t_s)[K(t_s - \varepsilon) \pm u(t_s - \varepsilon)] < 0 \tag{A.2-16}$$

and from GLC condition we have

$$(-1)^q \beta(t_s) > 0 \tag{A.2-17}$$

Multiplying (A.2-16) by (A.2-17) we get

$$(-1)^{q+r} \varepsilon^{2q} \beta^2(t_s)[K(t_s - \varepsilon) \pm u(t_s - \varepsilon)] < 0 \tag{A.2-18}$$

Since $|u(t)| \leq K(t)$ for all $t \in [t_0, t_f]$ and the singular arc is assumed to be interior almost everywhere, the bracketed quantity (A.2-18) regardless of $\pm$ signs is positive, also $\varepsilon^{2q}\beta^2(t_s) > 0$. Therefore

$$(-1)^{q+r} < 0 \tag{A.2-19}$$

which implies $q+r$ is odd.

APPENDIX B

B.1 Linear Case

Assume t_s^i to be a nominal switching time, we try to find $\frac{\partial TEF}{\partial t_s}$ due to the choice of t_s^i. For this purpose we calculate

$$\begin{bmatrix} \Delta x(t_f) \\ \Delta\lambda(t_f) \end{bmatrix} = [\Theta(t_s^i)] \begin{bmatrix} \Delta x(t_0) \\ \Delta\lambda(t_0) \end{bmatrix} + [\Psi(t_f,t_s^i)][\Delta\Phi(t_s)] \begin{bmatrix} x(t_0) \\ \lambda(t_0) \end{bmatrix}$$

$$+ [\Delta\Psi(t_s)] \begin{bmatrix} x(t_s^i) \\ \lambda(t_s^i) \end{bmatrix} + [\Psi(t_f,t_s^i)][\Delta\eta(t_s)] \qquad \text{(B.1-2)}$$

where

$$\Theta(t_s^i) = \Psi(t_f,t_s^i)\Phi(t_s^i,t_0) \qquad \text{(B.1-3)}$$

For simplicity we drop t_s^i, t_f and t_0 from transition matrices

$$\Delta\Phi(t_s) \simeq [\Phi] \begin{bmatrix} A & -CR^{-1}C^T \\ -Q & -A^T \end{bmatrix} \Delta t_s = \bar{D}\Delta t_s \qquad \text{(B.1-4)}$$

$$\Delta\Psi(t_s) \simeq -[\Psi] \begin{bmatrix} A+BM & BN-CR^{-1}C^T \\ -Q & -A^T \end{bmatrix} \Delta t_s = \bar{E}\Delta t_s \qquad \text{(B.1-5)}$$

$$\Delta\eta(t_s) = \begin{bmatrix} \pm BK \\ 0 \end{bmatrix} \Delta t_s \qquad \text{(B.1-6)}$$

Since $x(t_0)$ is fixed $\Delta x(t_0) = 0$, and to find $\Delta\lambda(t_0)$, the $2n \times 2n$ matrices Φ, Ψ, Θ, $\bar{D}$, $\bar{E}$ are partitioned to $n \times n$ matrices as the following.

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}, \quad \Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix}, \quad \Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$$

$$\bar{D} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \text{ and } \bar{E} = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}$$

From (3.1-3) we can obtain

$$\lambda(t_s) = \bar{K}x(t_s) \tag{B.1-7}$$

where $\bar{K}$ is an $n \times n$ matrix. So, we can have

$$\Delta\lambda(t_s) = \bar{K}\Delta x(t_s) \tag{B.1-8}$$

From (B.1-8) and (3.1-17) $\Delta\lambda(t_0)$ is obtained as

$$\Delta\lambda(t_0) = \bar{B}\,\Delta t_s \tag{B.1-9}$$

where

$$\bar{B} = [\Phi_{22} - \bar{K}\Phi_{12}]^{-1}[\bar{K}D_{11}x(t_0) + \bar{K}D_{12}\lambda(t_0) + \bar{K}BK - \bar{D}_{21}x(t_0) - D_{22}\lambda(t_0)] \tag{B.1-10}$$

Since the inverse in (B.1-10) always exists, we will have

$$\Delta x(t_f) \simeq \bar{L}\,\Delta t_s \tag{B.1-11}$$

$$\Delta\lambda(t_f) \simeq \bar{M}\,\Delta t_s \tag{B.1-12}$$

where

$$\bar{L} = [\Theta_{12}\bar{B} + (\Psi_{11}\bar{D}_{11} + \Psi_{12}\bar{D}_{21})x(t_0) + (\Psi_{11}\bar{D}_{12} + \Psi_{12}\bar{D}_{22})\lambda(t_0)$$

$$+ \bar{E}_{11}x(t_s^i) + \bar{E}_{12}\lambda(t_s^i) \pm \Psi_{11}BK] \tag{B.1-13}$$

$$\bar{M} = [\Theta_{22}\bar{B} + (\Psi_{21}\bar{D}_{11} + \Psi_{22}\bar{D}_{21})x(t_0) + (\Psi_{21}\bar{D}_{12} + \Psi_{12}\bar{D}_{22})\lambda(t_0)$$

$$+ E_{21}x(t_s^i) + \bar{E}_{22}\lambda(t_s^i) + \Psi_{21}BK \tag{B.1-14}$$

From (B.1-1), (B.1-11) and (B.1-12) we will get

$$\Delta TEF^i = (\bar{M} - S\bar{L})\big|_{t_s^i}\,\Delta t_s \tag{B.1-15}$$

So, when Δt approaches zero we will have

$$\left.\frac{\partial TEF}{\partial t_s}\right|_{t_s^i} = \bar{M} - S\bar{L}\big|_{t_s^i}$$

By having TEF and $\frac{\partial TEF}{\partial t_s}$ for each switching time t_s^i we can use Newton's method to find t_s at which $\|TEF\| = 0$, $\Delta x(t_f)$ and $\Delta\lambda(t_f)$ can be expressed in terms of Δt_s to the second order in which case by iterative methods we can get the optimal t_s where

$\frac{\partial}{\partial t_s} \|TEF\| = 0$ but it will be tedious.

B.2 Non-Linear Case

In this section we are trying to find an expression for $\frac{\partial TEF}{\partial t_s}$.
The solution to (3.2-12) and (3.2-13) is

$$\begin{bmatrix} \delta x(t_s^-) \\ \delta\lambda(t_s^-) \end{bmatrix} = [\Phi(t_s,t_0)] \begin{bmatrix} \delta x(t_0) \\ \delta\lambda(t_0) \end{bmatrix} \tag{B.2-1}$$

where

$$[\dot{\Phi}(t_s,t_0)] = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial \lambda} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial \lambda} \end{bmatrix} [\Phi(t_s,t_0)] \tag{B.2-2}$$

$$\Phi(t_0,t_0) = I \tag{B.2-3}$$

and the solution to (3.2-14) and (3.2-15) is

$$\begin{bmatrix} \delta x(t_f) \\ \delta\lambda(t_f) \end{bmatrix} = [\Psi(t_f,t_s)] \begin{bmatrix} \delta x(t_s^+) \\ \delta\lambda(t_s^+) \end{bmatrix} \tag{B.2-4}$$

where

$$[\dot{\Psi}(t_f,t_s)] = \begin{bmatrix} \frac{\partial \tilde{f}}{\partial x} & \frac{\partial \tilde{f}}{\partial \lambda} \\ \frac{\partial \tilde{g}}{\partial x} & \frac{\partial \tilde{g}}{\partial \lambda} \end{bmatrix} [\Psi(t_f,t_s)] \tag{B.2-5}$$

$$\Psi(t_f,t_f) = I \tag{B.2-6}$$

In order to express $\delta x(t_f)$ and $\delta\lambda(t_f)$ we eliminate $\Delta x(t_s)$ between (3.2-8) and (3.2-10) and $\Delta\lambda(t_s)$ between (3.2-9) and (3.2-11) we get

$$\begin{bmatrix} \delta x(t_s^+) \\ \delta\lambda(t_s^+) \end{bmatrix} = \begin{bmatrix} \dot{x}(t_s^-) + \dot{x}(t_s^+) \\ \dot{\lambda}(t_s^-) + \dot{\lambda}(t_s^+) \end{bmatrix} \Delta t_s + \begin{bmatrix} \delta x(t_s^-) \\ \delta\lambda(t_s^-) \end{bmatrix} \tag{B.2-7}$$

From (B.2-7) and (B.2-1) we have

$$\begin{bmatrix} \delta x(t_s^+) \\ \delta\lambda(t_s^+) \end{bmatrix} = \begin{bmatrix} \dot{x}(t_s^-) + \dot{x}(t_s^+) \\ \dot{\lambda}(t_s^-) + \dot{\lambda}(t_s^+) \end{bmatrix} \Delta t_s + [\varphi(t_s,t_0)] \begin{bmatrix} 0 \\ \delta\lambda(t_0) \end{bmatrix} \tag{B.2-8}$$

and from (B.2-8) and (B.2-4) we obtain

$$\begin{bmatrix} \delta x(t_f) \\ \delta\lambda(t_f) \end{bmatrix} = [\alpha(t_s)]\Delta t_s + \theta(t_s) \begin{bmatrix} 0 \\ \delta\lambda(t_0) \end{bmatrix} \tag{B.2-9}$$

where

$$\alpha(t_s) = [\Psi(t_f,t_s)] \begin{bmatrix} \dot{x}(t_s^-) + \dot{x}(t_s^+) \\ \dot{\lambda}(t_s^-) + \dot{\lambda}(t_s^+) \end{bmatrix} \tag{B.2-10}$$

$$\theta(t_s) = [\Psi(t_f,t_s)][\Phi(t_s,t_0)] \tag{B.2-11}$$

By linearizing (3.1-3) around a nominal trajectory we will get

$$\begin{bmatrix} (H_u)_x & (H_u)_\lambda \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ (H_u^{(n-1)})_x & (H_u^{(n-1)})_\lambda \end{bmatrix} \begin{bmatrix} \delta x(t_s^-) \\ \delta\lambda(t_s^-) \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ -H_u^{(n)} \end{bmatrix} \Delta t_s \tag{B.2-12}$$

From (B.2-1) and (B.2-12) we obtain

$$\begin{bmatrix} 0 \\ \delta\lambda(t_0) \end{bmatrix} = \gamma(t_s)\ \Delta t_s \tag{B.2-13}$$

where

$$\gamma(t_s) = [\beta(t_s)]^{-1} \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ -H_u^{(n)} \end{bmatrix} \tag{B.2-14}$$

$$\beta(t_s) = \begin{bmatrix} (H_u)_x & (H_u)_\lambda \\ \vdots & \vdots \\ H_u^{n-1})_x & (H_u^{n-1})_\lambda \end{bmatrix} [\varphi(t_s, t_0)] \tag{B.2-15}$$

From (B.2-13) and (B.2-9) we have

$$\begin{bmatrix} \delta x(t_f) \\ \delta\lambda(t_f) \end{bmatrix} = [\alpha(t_s) + \theta(t_s)\gamma(t_s)]\Delta t_s = \mu(t_s)\Delta t_s \tag{B.2-16}$$

From (3.2-18)

$$\Delta TEF = \delta\lambda(t_f) - \frac{\partial^2 h}{\partial x^2}\, \delta x(t_f) + H.O.T. \tag{B.2-17}$$

Now partitioning $\mu(t_s) = \begin{bmatrix} \mu_1(t_s) \\ \mu_2(t_s) \end{bmatrix}$ where $\mu_1(t_s)$ and $\mu_2(t_s)$ are

$$\Delta TEF = \mu_1(t_s)\, \Delta t_s - \frac{\partial^2 h}{\partial x^2}\, \mu_2(t_s)\, \Delta t_s + 0(\Delta t_s)^2 \tag{B.2-18}$$

$$\left.\frac{\partial TEF}{\partial\, t_s}\right|_{t_s^i} = \frac{\Delta TEF}{\Delta t_s \to 0} = \mu_1(t_s^i) - \frac{\partial^2 h}{\partial x^2}\, \mu_2(t_s^i) \tag{B.2-19}$$

Therefore, similar to the linear case we can find an iterative relationship for the switching time.

APPENDIX C

In this appendix first we express TEF_i as a function of $\lambda^i(t_0)$ and t_s^i. For this purpose we find the final states and costates of the system from (3.1-8) and (3.1-12) as

$$\begin{bmatrix} x^i(t_f) \\ \lambda^i(t_f) \end{bmatrix} = [\Theta(t_s^i)] \begin{bmatrix} x(t_0) \\ \lambda(t_0) \end{bmatrix} + [\gamma(t_s^i)] \tag{C.1}$$

where $\Theta(t_s^i)$ is defined in Appendix B and

$$\gamma(t_s^i) = \Psi(t_f, t_s^i)\eta(t_s^i) = \begin{bmatrix} \gamma_1(t_s^i) \\ \gamma_2(t_s^i) \end{bmatrix} \tag{C.2}$$

γ_1 and γ_2 are $n \times 1$ vectors as a function of t_s^i.

By definition

$$n_i^2 = [\lambda^i(t_f) - Sx^i(t_f)]^T[\lambda^i(t_f) - Sx^i(t_f)] \tag{C.3}$$

From (C.1) we have

$$\begin{cases} x(t_f) = \Theta_{11}x(t_0) + \Theta_{12}\lambda(t_0) + \gamma_1 & \text{(C.4)} \\ \lambda(t_f) = \Theta_{21}x(t_0) + \Theta_{22}\lambda(t_0) + \gamma_2 & \text{(C.5)} \end{cases}$$

and from (C.2) - (C.5) we will obtain

$$n_i^2 = \bar{P}(t_s^i) + 2\bar{Q}(t_s^i)\lambda^i(t_0) + \lambda^{i^T}(t_0)\bar{R}(t_s^i)\lambda^i(t_0) \qquad (C.6)$$

where

$$\bar{Q}(t_s^i) = [(\theta_{21} - S\theta_{11})x(t_0) + (\gamma_2 - S\gamma_1)]^T(\theta_{22} - S\theta_{12}) \qquad (C.7)$$

$$\bar{R}(t_s^i) = [\theta_{22} - S\theta_{12})^T(\theta_{22} - S\theta_{12})] \qquad (C.8)$$

$$\bar{P}(t_s^i) = (\theta_{21} - S\theta_{12})x(t_0) + (\gamma_2 - S\gamma_1)]^T[(\theta_{21} - S\theta_{11})x(t_0) + (\gamma_2 - S\gamma_1)] \qquad (C.9)$$

Since Condition (3.1-3) is linear in states and costates it can be expressed as

$$[V(t_s^i)]\begin{bmatrix} x(t_s^i) \\ \lambda(t_s^i) \end{bmatrix} = 0 \qquad (C.10)$$

where $V(s)$ is an $m \times 2n$ matrix is a function of system parameters.

From (C.10) and (3.1-18) we will obtain

$$[W(t_s^i)]\begin{bmatrix} x(t_0) \\ \lambda(t_0) \end{bmatrix} = T(t_s^i)] \qquad (C.11)$$

where

$$[W(t_s^i)] = \begin{bmatrix} W_1(t_s^i) \\ W_2(t_s^i) \end{bmatrix} = [V(t_s^i)][\Phi(t_s^i,t_0)] \qquad (C.12)$$

and

$$T(t_s^i) = [V(t_s^i)][\eta(t_s^i)] \qquad (C.13)$$

W_1 and W_2 are $m \times n$ matrices and T is an m vector. So, we will have

$$W_2(t_s^i)\lambda(t_0) = T(t_s^i) - W_1(t_s^i)x(t_0) \qquad (C.14)$$

Now (C.6) can be minimized with respect to $\lambda(t_0)$ subject to (C.14) at any t_s^i.

For computation of $\frac{\partial}{\partial t_s}[\min_{\lambda(t_0)} h]$ at each t_s^i we differentiate (C.4) and (C.5) with respect to t_s

$$\begin{cases} \dfrac{\partial x(t_f)}{\partial t_s} = \dfrac{\partial \theta_{11}}{\partial t_s} x(t_0) + \dfrac{\partial \theta_{12}}{\partial t_s} \lambda(t_0) + \theta_{12} \dfrac{\partial \lambda(t_0)}{\partial t_x} + \dfrac{\partial V_1}{\partial t_s} \\ \dfrac{\partial \lambda(t_f)}{\partial t_s} = \dfrac{\partial \theta_{21}}{\partial t_s} x(t_0) + \dfrac{\partial \theta_{22}}{\partial t_s} \lambda(t_0) + \theta_{22} \dfrac{\partial \lambda(t_0)}{\partial t_s} + \dfrac{\partial V_2}{\partial t_s} \end{cases}$$

at each switching time t_s^i $\lambda^i(t_0)$. $\frac{\partial \lambda^i(t_0)}{\partial t_s}$ and $\min_{\lambda(t_0)} h(t_s^i)$ are obtained so $\frac{\partial}{\partial t_s}[\min h(t_s)]$ can be computed, then by a Newton iterative relationship the optimal switching time is reached.

END